Sommerfeld Theory of Metals

Pauli exclusion principle, part of quantum theory, changed the understanding of electrons in metals and solved several important problems from Drude theory.

It all has to do with the velocity distribution.

For a classical gas, \( v \) is distributed according to Maxwell-Boltzmann distribution.

The probability for speed \( v \) is:

\[
f_B(v) = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}}
\]

This looks like:

\[
f_B \sim n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} \quad e^0 = 1
\]
\( \mathbf{v} \) is 3-D with components \( v_x, v_y, v_z \)

The actual number of \( \Theta \) particles per unit volume with speeds between \( v \) and \( v + dv \) is

\[
n(v)dv = \frac{\Theta}{4\pi} f_8(v) dv
\]

in spherical coords., \( dv = 4\pi v^2 dv \)

The factor of \( 4\pi v^2 dv \) results in:

\( v^2 = 0 \)
With advent of quantum theory and Pauli exclusion, it was found that electrons should follow Fermi-Dirac distribution:

\[ f_{F-D}(v) = \frac{(m/\pi)^{3/2}}{4 \pi^3} \frac{1}{\exp \left( \frac{\pi v^2}{2 kT} - \frac{1}{kT} \right) + 1} \]

The total number of electrons per unit volume is

\[ n = \int f_{F-D}(v) \, dv \]
Ground State Properties of Electron Gas

\[-\frac{\hbar^2}{2m} \nabla^2 \psi (\vec{r}) = E \psi (\vec{r})\]

how are electrons confined?

Born-von Karman boundary conditions

\[
\psi (x, y, z + L) = \psi (x, y, z) \\
\psi (x, y + L, z) = \psi (x, y, z) \\
\psi (x + L, y, z) = \psi (x, y, z)
\]

Solution is

\[
\psi_h (\vec{r}) = \frac{1}{\sqrt{V}} e^{i \vec{h} \cdot \vec{r}}
\]

\(\vec{h}\) is the wave vector

\[
E(\vec{h}) = \frac{\hbar^2 \vec{h}^2}{2m}
\]

\(\psi\) is normalized to 1 in volume \(V\)

\(\psi\) is an eigenstate of the momentum operator

\[
\hat{p}_\alpha = \frac{\hbar}{i} \frac{\partial}{\partial x^\alpha} = \frac{\hbar}{i} \nabla
\]
i.e. \[ \hat{\rho} | \psi \rangle = \hat{\rho} \psi \]

\[
\left( \frac{\hbar}{i} \frac{\partial}{\partial \tau} \right) (e^{i \hbar \tau \hat{\rho}}) = \left( \frac{\hbar}{i} \frac{\partial}{\partial \tau} \right) (e^{i \hbar \tau \hat{\rho}})
\]

\[ \Rightarrow \hat{\rho} = \hbar \tau \; \; \; \text{eigenstate of } \hat{\rho} \]

\[ \tau = \frac{\hbar \tau}{m} \]

\[ \varepsilon = \frac{p^2}{2m} = \frac{1}{2} m v^2 \]

\[ e^{i \hbar \tau \hat{\rho}} \] is a plane wave with wavelength

\[ \lambda = \frac{2 \pi}{k} \; \; \; \text{spatial period of the wavefunction} \]

boundary conditions

\[ e^{i \hbar \tau \hat{\rho}} (\hat{\rho} + L_x \hat{\rho}) = e^{i \hbar \tau \hat{\rho}} \]

\[ \Rightarrow \hat{e} = e \]

\[ \Rightarrow e^{i k_x L_x} = 1 \]

Similarly \[ e^{i k_y L_y} = 1 \]

\[ + e^{i k_z L_z} = 1 \]
\( e^{i k_x L_x} = 1 \quad \Rightarrow \quad k_x = \frac{2\pi n_x}{L_x} \)

Similarly \( \Rightarrow \quad k_y = \frac{2\pi n_y}{L_y} \)

\( k_z = \frac{2\pi n_z}{L_z} \quad \text{integers} \)

\( \Rightarrow \) allowed wave vectors are integral multiples

\( \frac{2\pi}{L_x} \) etc.

\[ h_x \quad h_y \quad k\text{-space} \]

If we have a \( k\text{-space} \) region of size \( \Omega \),

the \# of \( k\text{-space} \) pts. is

\[ N_k = \frac{\Omega}{(2\pi)^3 L_x L_y L_z} = \frac{\Omega V}{(2\pi)^3} \]

\( \Rightarrow \) density of levels \( = \frac{V}{(2\pi)^3} \)