731 Homework #2 70 pts total

Due September 18, 2006

(10) 1. derive equation 1.40 and 1.41

(30) 2. A&M, problem 1.4 "Helicon Waves"

(30) 3. A&M, problem 1.5 "Surface Plasmons"

HW #2 [Total 70 pts]

[redacted]

Solution schematic

1) derive Eq. 1.40 and 1.41

(10 pts)

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

$$n = \frac{1}{\frac{4}{3}\pi r_s^3}$$

insert n into expression for ω_p^2

take square root

divide by 2π

$$\Rightarrow \quad \nu_p = \frac{\sqrt{\frac{3e^2}{m a_0^3}}}{2\pi} \left(\frac{r_s}{a_0}\right)^{-3/2}$$

$$e = 4.8032 \times 10^{-10} \text{ esu}$$

$$a_0 = .529 \times 10^{-8} \text{ cm}$$

$$m = 9.1094 \times 10^{-28} \text{ g}$$

plug in e, m, a_0

$$\Rightarrow \quad \nu_p = 11.4 \times \left(\frac{r_s}{a_0}\right)^{-3/2} \times 10^{15} \text{ Hz}$$

$$\lambda_p = \frac{c}{\nu_p} = .26 \left(\frac{r_s}{a_0}\right)^{3/2} \times 10^{3} \text{ Å}$$

HW#2

$\boxed{2, \text{ AM } 1.4}$ Helicon Waves (30 pts)

$\boxed{\text{Solution schematic}}$

(a) (8 pts) Start with

$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{\vec{p}}{mc} \times \vec{H}\right) - \frac{\vec{p}}{\tau}$$

$$\frac{d\vec{p}}{dt} = -i\omega\vec{p}$$

get x and y components

$$-i\omega p_x = -eE_x - \omega_c p_y - \frac{p_x}{\tau}$$

$$-i\omega p_y = -eE_y + \omega_c p_x - \frac{p_y}{\tau}$$

$$p_z = 0$$

use $E_y = \pm i E_x$

$\Rightarrow$ plug in and solve to find

$$p_y = \pm i p_x$$

then solve for p_x in terms of E_x

$$\Rightarrow \quad p_x = \frac{-e\tau}{1 - i(\omega \mp \omega_c)\tau} E_x$$

Then $\dot{\vec{j}} = -\frac{ne}{m}\vec{p}$ ②

$\Rightarrow \quad j_x = \dfrac{\left[\frac{ne^2\tau}{m}\right]}{1-i(\omega\mp\omega_c)\tau} E_x$ $\leftarrow \sigma_0$

and $j_y = \pm i j_x \qquad j_z = 0$

(b) (8 pts) $\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{H} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{d\vec{H}}{dt}$

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \Rightarrow \boxed{-\nabla^2\vec{E} = \frac{\omega^2}{c^2}\left(1+\frac{4\pi i\sigma}{\omega}\right)\vec{E}}$

let $E_x = E_0 e^{i(kz-\omega t)}$

use $j_x = \dfrac{\sigma_0}{1-i(\omega\mp\omega_c)\tau} E_x$ with 2 components (x & y) of

get $-\frac{d^2}{dz^2}E_x = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega}\,\frac{1}{\omega\mp\omega_c+\frac{i}{\tau}}\right)E_x$

plug in E_x

$\Rightarrow k^2c^2 = \epsilon\omega^2 \qquad \epsilon = 1 - \frac{\omega_p^2}{\omega}\left(\frac{1}{\omega\mp\omega_c+i/\tau}\right)$

same for $E_y = E_0' e^{i(kz - \omega t)}$ 3

$$E_y = \frac{E_0'}{E_0} E_x$$

$$\Rightarrow \quad \frac{(\pm i)}{\left(\frac{E_0'}{E_0}\right)} = 1 \quad \Rightarrow \quad \frac{E_0'}{E_0} = \pm i$$

(c) (7 pts) $\omega_c \tau \gg 1 \Rightarrow$

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega(\omega_c - \omega)}$$

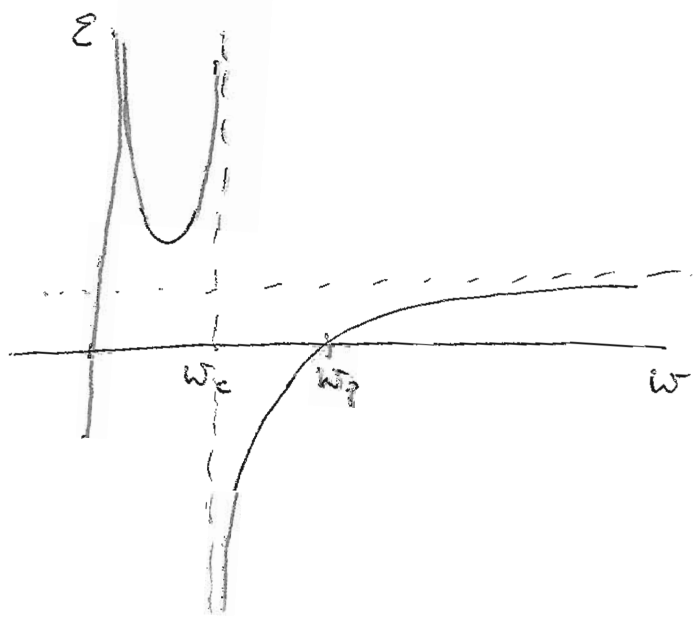
at $\omega = \omega_p$

$$\varepsilon = 1 + \frac{\omega_p^2}{\omega_p(\omega_c - \omega_p)}$$

$$= \frac{\omega_p(\omega_c - \omega_p) + \omega_p^2}{\omega_p(\omega_c - \omega_p)}$$

$$\varepsilon = \frac{\omega_p \omega_c}{\omega_p(\omega_c - \omega_p)}$$

$$\varepsilon = \frac{\omega_c}{\omega_c - \omega_p}$$



but for high field $\omega_p \gg \omega_c$

$$\Rightarrow \quad \varepsilon = 0$$

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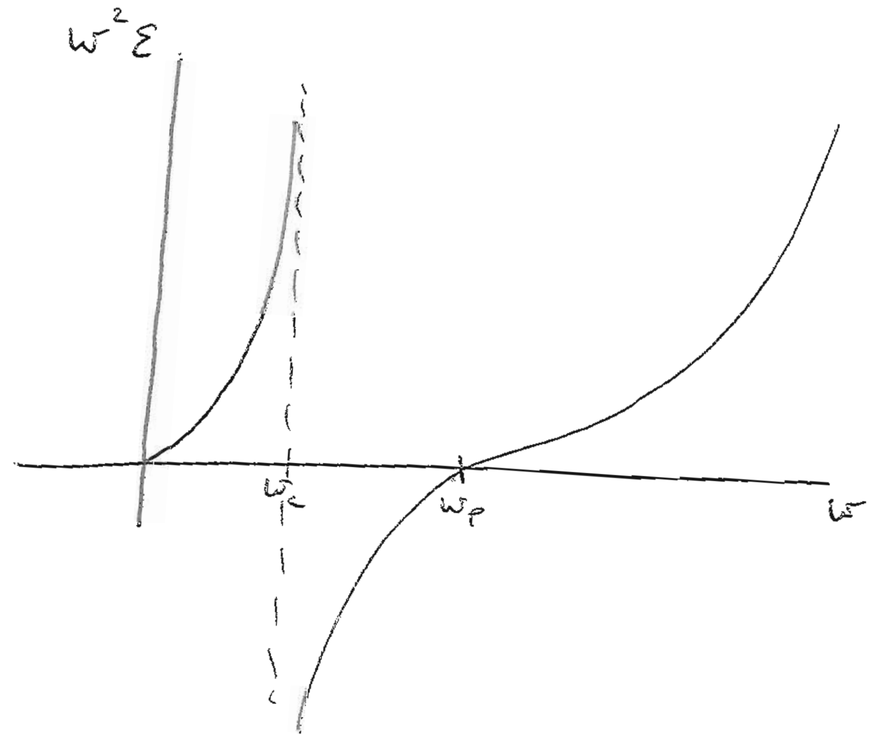
$$\omega^2 \varepsilon = \omega^2 + \frac{\omega\, \omega_p^2}{(\omega_c - \omega)}$$



$\Rightarrow$ for $\omega < \omega_c$ or $\omega > \omega_p$

$$k^2 c^2 = \epsilon\, \omega^2 = \text{any positive value}$$

$\Rightarrow$ always a solution for k

$$k = \frac{\omega}{c}\sqrt{\varepsilon}$$

(d) (7 pts) $\omega << \omega_c$ low frequency solution (5)

and for $\omega_c \tau >> 1$

$$k^2c^2 = \omega^2 + \frac{\omega\,\omega_p^2}{\omega_c}$$

$$k^2c^2 = \frac{\omega\,\omega_p^2}{\omega_c}\left(1 + \frac{\omega\,\omega_c}{\omega_p^2}\right)$$

$$k^2c^2 \simeq \frac{\omega\,\omega_p^2}{\omega_c}$$

$$\omega = \frac{\omega_c}{\omega_p^2} k^2c^2$$

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for $\lambda = 1\text{ cm}$

$\cancel{B}$ $H = 10\text{ kG} = 10^4\text{ G}$

$$\omega = \frac{\frac{eH}{mc}}{\frac{4\pi n e^2}{m}}\left(\frac{2\pi}{\lambda}\right)^2 c^2 = \frac{\pi H c}{\lambda^2 n e}$$

use typical $n = 5\times10^{22}/\text{cm}^3$ $\Rightarrow$ $\omega = \dfrac{\pi(10^4\text{ G})(3\times10^{10}\text{ cm/s})}{(1\text{ cm})^2(5\times10^{22}/\text{cm}^3)(4.8\times10^{-10}\text{ esu})}$

$= 39$ rad/s

$\Rightarrow \nu \simeq 6.25$ Hz

~~$\omega = 2222$ rad/s~~

~~$\omega = 196$ Hz~~

$\Rightarrow f = 6.25$ Hz

HW #2 ~~[illegible]~~

Solution

$\boxed{\text{3. AM 1.5 Surface Plasmons}}$ (30pts)

$$\epsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} \qquad \sigma_0 = \frac{ne^2\tau}{m}$$

$\vec{E}_{\parallel}$ continuous

$(\epsilon \vec{E})_{\perp}$ continuous

(10 pts) (a) Find 3 equations relating q, K, K' as functions of ω

Note: we assume $(\epsilon \vec{E})_{\perp}$ continuous

$$\Rightarrow \boxed{\begin{aligned} q^2 - K^2 &= \frac{\omega^2}{c^2}\epsilon \\ q^2 - K'^2 &= \frac{\omega^2}{c^2} \\ K &= -K'\epsilon \end{aligned}}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} = \frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}$$

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{d}{dx}E_x + \frac{d}{dy}E_y + \frac{d}{dz}E_z = 0$$

$$A iq\, e^{iqx} e^{-kz} + 0 - kB\, e^{iqx-kz} = 0$$

$$\boxed{A iq = kB}$$

also

$$\boxed{C iq = -k' D}$$

$$\frac{A iq}{C iq} = \frac{kB}{-k'D}$$

but $A = C$ and $\epsilon B = D$

$$\Rightarrow \quad 1 = \frac{k}{-k'\epsilon}$$

$$\Rightarrow \quad \boxed{k = -k'\epsilon}$$

$z > 0$

$$-\nabla^2 E_x = \frac{\omega^2}{c^2} \epsilon E_x$$

$$\Rightarrow \quad (iq)^2 E_x + (-K)^2 E_z = K^2 - q^2 E_x$$

$$\Rightarrow \quad \boxed{q^2 - K^2 = \frac{\omega^2}{c^2} \epsilon}$$

$z < 0$

$$-\nabla^2 E_x = \frac{\omega^2}{c^2} E_x$$

$$\Rightarrow \quad \boxed{q^2 - K'^2 = \frac{\omega^2}{c^2}}$$

b) (10 pts) assuming $\omega\tau \gg 1$, plot q^2c^2 vs. ω^2

$$\omega\tau \gg 1 \Rightarrow \epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

$$\star^1 \quad q^2c^2 - K^2c^2 = \omega^2\epsilon$$

$$q^2c^2 - K'^2c^2 = \omega^2$$

$$\rightarrow q^2c^2 - \frac{K^2}{\epsilon^2}c^2 = \omega^2$$

$$\star^2 \quad \rightarrow q^2c^2\epsilon^2 - K^2c^2 = \omega^2\epsilon^2$$

$$\star^1 - \star^2 = q^2c^2(1-\epsilon^2) = \omega^2(\epsilon - \epsilon^2)$$

$$\rightarrow q^2c^2 = \frac{\omega^2\epsilon(1-\epsilon)}{(1+\epsilon)(1-\epsilon)}$$

$$q^2c^2 = \frac{\omega^2\epsilon}{1+\epsilon}$$

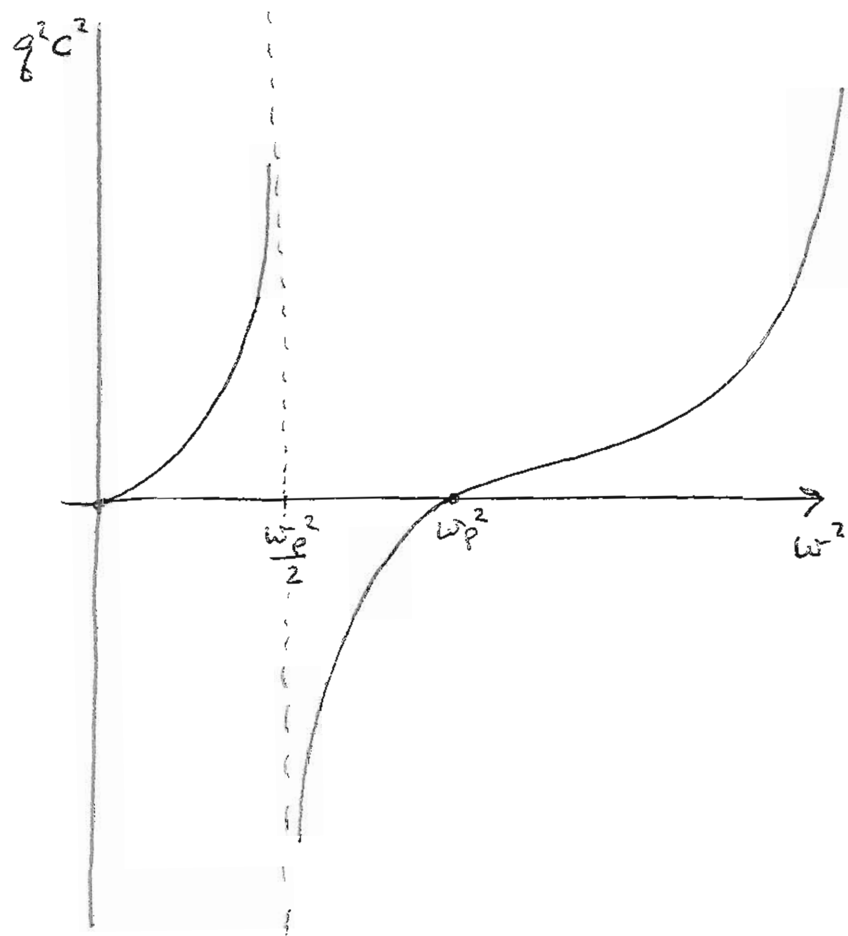
$$= \frac{\omega^2(1 - \omega_p^2/\omega^2)}{2 - \omega_p^2/\omega^2}$$

$$\boxed{q^2c^2 = \frac{\omega^2(\omega^2 - \omega_p^2)}{2\omega^2 - \omega_p^2}}$$

$$q^2 c^2 = \frac{\omega^2 (\omega^2 - \omega_p^2)}{2\omega^2 - \omega_p^2}$$

at $\omega = 0 \Rightarrow q^2 c^2 = \frac{0\,(-\omega_p^2)}{-\omega_p^2} = 0$

at $\omega = \omega_p \Rightarrow q^2 c^2 = \frac{\omega_p^2\,(0)}{\omega_p^2} = 0$



(c)
(10 pts)

limit $qc \gg \omega$

or $\omega \ll qc$

or $\omega^2 \ll q^2c^2$

show there is a solution at $\omega = \omega_p/\sqrt{2}$

show that the wave is confined to the surface,

Describe its polarization

$\rightarrow$ surface plasmon

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from the graph in (b), there is a solution

at $\boxed{\omega^2 = \frac{\omega_p^2}{2} - \delta}$ in which $q^2c^2 \gg \omega^2$

$$\Rightarrow \boxed{\omega \simeq \frac{\omega_p}{\sqrt{2}}}$$

$$\Rightarrow \epsilon = 1 - \frac{\omega_p^2}{\omega_p^2/2} = 1 - \frac{1}{1/2}$$

$$= 1 - 2$$

$$\boxed{\epsilon = -1}$$

(c) <u>cont.</u>

$$\epsilon = -1$$

$$\Rightarrow \quad \underline{K = K'}$$

then

$$q^2 c^2 - K^2 c^2 = \omega^2 (-1)$$

and

$$q^2 c^2 - K'^2 c^2 = \omega^2$$

~~but $q^2 c^2 >> \omega^2$~~

$$\Rightarrow \quad q^2 c^2 + \omega^2 = K^2 c^2$$

but $q^2 c^2 >> \omega^2$

$$\Rightarrow \quad q^2 c^2 \simeq K^2 c^2$$

at $\omega^2 = \frac{\omega_p^2}{2} - \delta$, $q^2 c^2$ is $>> 0$

$\Rightarrow \quad K^2 c^2 >> 0$ approaching ∞

$$\Rightarrow \quad K^2 c^2 \sim \infty$$

$\Rightarrow \quad K \sim \infty$ or very large

e^{-Kz} with K large $\Rightarrow$ surface confinement similarly since

$$K' = K$$

$e^{K'z}$ with large K' and $z(-)$

$\Rightarrow$ surface confinement

<u>polarization</u>

$$z = 0^+ \quad \frac{E_z}{E_x} = \frac{B}{A} \qquad H_y = H_0 e^{iqx} e^{-Kz}$$

$$\vec{H} = H_y \hat{y}$$

no x or z

must use

$$\frac{i\omega}{c} \vec{\nabla} \times \vec{H} = \frac{\omega^2}{c^2} \epsilon \vec{E}$$

$$\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ H_x & H_y & H_z \end{vmatrix} = \left(\cancel{\frac{d}{dy} H_z} - \frac{d}{dz} H_y\right)\hat{x} + \left(\frac{d}{dx} H_y - \cancel{\frac{d}{dy} H_x}\right)\hat{z}$$

$$= -\frac{d}{dz} H_y \hat{x} + \frac{d}{dx} H_y \hat{z}$$

$$\frac{i\omega}{c}\left(-\frac{d}{dz} H_y\right) = \frac{\omega^2}{c^2} \epsilon E_x$$

$$\boxed{\frac{i\omega K}{c} H_y = \frac{\omega^2}{c^2} \epsilon E_x}$$

and $$\frac{i\omega}{c}\left(\frac{d}{dx} H_y\right) = \frac{\omega^2}{c^2} \epsilon E_z$$

$$\boxed{-\frac{q\omega}{c} H_y = \frac{\omega^2}{c^2} \epsilon E_z}$$

$$\Rightarrow \quad \frac{E_z}{E_x} = \frac{-q\omega/c}{\frac{i\omega K}{c}} = -\frac{q}{iK} = \boxed{i\frac{q}{K}}$$

$\underline{z > 0^-}$ Same except

$$\frac{E_z}{E_x} = \frac{-iq}{K'}$$

$\underline{z=0^+}$:

$$\Rightarrow \quad \frac{E_z}{E_x} = i\sqrt{\frac{q^2c^2}{K^2c^2}}$$

$$= i\sqrt{\frac{\frac{\omega^2(\omega^2-\omega_p^2)}{2\omega^2-\omega_p^2}}{\frac{\omega^2(2\omega^2-\omega_p^2)}{2\omega^2-\omega_p^2} + \omega^2(\omega^2-\omega_p^2)}}$$

$$= i\sqrt{\frac{\omega^4-\omega^2\omega_p^2}{3\omega^4-2\omega^2\omega_p^2}}$$

$$= i\sqrt{\frac{\omega^2-\omega_p^2}{3\omega^2-2\omega_p^2}} \qquad \omega^2 = \frac{\omega_p^2}{2}$$

$$= i\sqrt{\frac{-\omega_p^2/2}{-\frac{1}{2}\omega_p^2}}$$

$$\frac{E_z}{E_x} = i$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi/2} = 0 + i$$

$$\Rightarrow \quad \frac{E_z}{E_x} = e^{i\frac{\pi}{2}} = \frac{B}{A}$$

$\Rightarrow$ 90° out of phase

(circular)

$\Rightarrow$ polarization

since

$$E_x = A e^{iqx} e^{-kz} \qquad E_z = A e^{i(qx + \frac{\pi}{2})} e^{-kz}$$

$$z = 0^- : \quad \frac{E_z}{E_x} = -i\frac{q}{K'} = -i\frac{q}{K}$$

$$\Rightarrow \quad \frac{E_z}{E_x} = -i$$

$$\Rightarrow \quad \frac{E_z}{E_x} = e^{-i\frac{\pi}{2}}$$

$\Rightarrow$ 90° out of phase $\Rightarrow$ circular polarization