731 Homework #3

Due September 25, 2006

(30 pts) 1. A&M, problem 2.1 5 pts each part

(20 pts) 2. derive A&M Eq. 2.48 starting from 2.41, showing all the steps.

3. Density of levels

total (20 pts)

(10) a. derive A&M Eqs. 2.61, the density of levels, g(ε)

(5) b. explain why the form of 2.60 shows that g(ε) is the density of levels (see 2.62)

(5) c. derive A&M Eqs. 2.63

70 Total

Solutions

Homework # 3

1) problem 2.1 (A, M)
(30 pts)

number of k in k_F (

$$\frac{\pi k_F^2}{(2\pi/L)^2} = \frac{k_F^2 L}{4\pi}$$

a)

$$\boxed{n = k_F^2/2\pi}$$

$$N = 2\,\frac{k_F^2 L^2}{4\pi}$$

2 spin

b) $k_F = 2/r_s$

c) $N = \sum_{k,s} f(\varepsilon(\vec{k})) = 2 \sum_{k} f(\varepsilon(\vec{k}))$

$$N = \frac{L^2}{4\pi^2} \cdot 2 \int d\vec{k}\, f(\varepsilon) \qquad , \qquad \varepsilon = \frac{\hbar^2}{2m} k^2 \; ; \quad d\varepsilon = \frac{\hbar^2 k}{m} dk$$

$$n = \frac{\sum_{k,s} f(\varepsilon)}{L^2} = \int_0^\infty \frac{2\pi k\, dk}{2\pi^2} f(\varepsilon)$$

$$= \int_0^\infty d\varepsilon \cdot \frac{m}{\pi\hbar^2} f(\varepsilon) \equiv \int_0 d\varepsilon\, g(\varepsilon) f(\varepsilon)$$

$$\rightarrow \boxed{g(\varepsilon) = \begin{cases} m/\pi\hbar^2 , & \varepsilon > 0 \\ 0 & \varepsilon < 0 \end{cases}}$$

d) $g(\varepsilon) = \frac{m}{\pi\hbar^2}$; $d = 2$

$$n = \int_0^\infty d\varepsilon \, g(\varepsilon) f(\varepsilon)$$

$$n = \int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 g'(\mu) + \dots$$

$$g(\varepsilon) = \text{constant} \to g'(\mu) = g''(\mu) = \dots = 0$$

$$\left.\begin{aligned} &\Rightarrow n = \int_0^\mu g(\varepsilon)\, d\varepsilon = \int_0^\mu \frac{m}{\pi\hbar^2} d\varepsilon = \frac{m\mu}{\pi\hbar^2} \\ &\text{ground state: } \quad n = \frac{k_F^2}{2\pi} = \frac{1}{2}\frac{2m}{\hbar^2}\epsilon_F = \frac{m\epsilon_F}{\pi\hbar^2} \end{aligned}\right\} \mu = \epsilon_F$$

e) $$n = \int_0^\infty d\varepsilon \, g(\varepsilon) f(\varepsilon) = \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon \frac{1}{e^{(\varepsilon-\mu)/k_B T}+1}$$

$$n = \frac{m}{\pi\hbar^2}\left(\varepsilon - kT\ln\left(1 + e^{(\varepsilon-\mu)/k_B T}\right)\right)\Big|_0^\infty$$

$$\left.\begin{aligned} &n = \left(\mu + kT\ln\left(1+e^{-\mu/k_B T}\right)\right)\frac{m}{\hbar^2\pi} \\ &n = \frac{m\epsilon_F}{\pi\hbar^2} \end{aligned}\right\} \longrightarrow$$

$$\Rightarrow \epsilon_F = \left(\mu + kT\ln\left(1 + e^{-\mu/k_B}\right)\right)$$

f) $\mu/_{k_BT} >> 1$ $\quad e^{-\mu/k_BT} << 1$

$\ln(1+e^{-\mu/k_BT}) \simeq$

$$\rightarrow \mu + k_BT \left(\frac{e^{-\mu/k_BT}}{\underbrace{1+e^{-\mu/k_BT}}_{\approx 1}} \right) + \cdots + = \epsilon_F$$

$$\mu + k_BT\, e^{-\mu/k_BT} \approx \epsilon_F$$

$T \simeq 300\,k$; $\approx e^{-\cancel{17}0}$, very smal.

Singular point $\quad \varepsilon \to 0^- \quad g(\varepsilon) = 0$

$\qquad \varepsilon \to 0^+ \quad g(\varepsilon) = \frac{m}{\pi \hbar^2}$

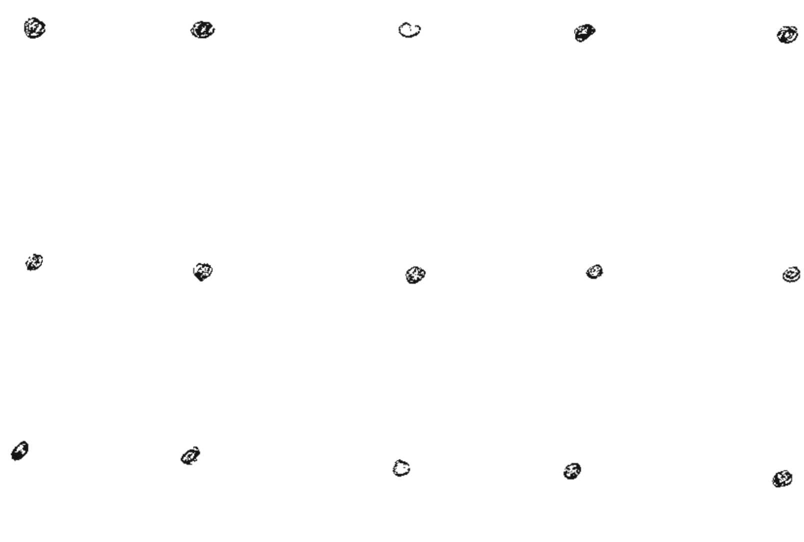
HW #3

Problem 2. (20 pts)

$N = 4$

ε_i

$N+1 = 5$



$$P_N\left(E_\alpha^{N+1} - \varepsilon_i\right) = \left[e^{-\left(E_\alpha^{N+1} - \varepsilon_i - F_N\right)/k_BT}\right] e^{F_{N+1}/k_BT} e^{-F_{N+1}/k_BT}$$

$$= e^{\left(\varepsilon_i - F_{N+1} + F_N\right)/k_BT}\, e^{\left(-E_\alpha/k_BT + F_{N+1}/k_BT\right)}$$

$$= e^{\left(\varepsilon_i - \mu\right)/k_BT}\, e^{-\left(E_\alpha^{N+1} - F_{N+1}\right)/k_BT}$$

$$\boxed{P_N\left(E_\alpha^{N+1} - \varepsilon_i\right) = e^{\left(\varepsilon_i - \mu\right)/k_BT} P_{N+1}\left(E_\alpha^{N+1}\right)} \quad (2.44)$$

$\mu = F_{N+1} - F_N$ = chemical potential

follow the steps

most important missing steps (worth 10 pt.) out of 20 total

3) (20 pts)

(10) a) $$\int \frac{d\vec{k}}{4\pi^3} F() = \int \frac{dk_x\, dk_y\, dk_z}{4\pi^3} F$$

$$= \frac{1}{\pi^2} \int dk \cdot k^2 \cdot F, \qquad \mathcal{E} = \frac{\hbar^2}{2m} k^2.$$

$$d\mathcal{E} = \frac{\hbar^2}{m} k\,dk, \qquad k^2 dk = \left(\frac{m}{\hbar}\right)^{3/2} \sqrt{2\epsilon}\, d\epsilon.$$

$$k = \sqrt{\frac{2m}{\hbar^2} \epsilon}$$

$$\int \frac{d\vec{k}}{4\pi^3} F = \int g(\mathcal{E}) F(\mathcal{E})\, d\mathcal{E} \quad ; \quad g = \frac{m}{\hbar^2 \pi} \sqrt{\frac{2m\epsilon}{\hbar^2}}.$$

(5) b) in the Book.

(5) c) $$g(\mathcal{E}) = \frac{m}{\pi^2 \hbar^2} \sqrt{\frac{2m\epsilon}{\hbar^2}} \quad ; \quad n = \frac{k_F^3}{3\pi^2}.$$

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} \rightarrow \frac{n}{\epsilon_F} \sqrt{1/\epsilon_F} = \frac{m}{\hbar^3 \pi^2} \frac{3}{2} \sqrt{2m}$$

$$g(\mathcal{E}) = \frac{3}{2}\, n/\epsilon_F \sqrt{\epsilon/\epsilon_F}.$$