PHY 731 Fall 2006 Midterm Exam (closed book) Name Key
Friday 10/6/2006

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Score: Multiple Choice: ________/ **27** ; Multi-part Problems: _______/ **60: Total Score** ________/ **87**

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Part I. Multiple Choice questions (please circle the correct answer) (**3 pts each**)

1. In a Drude metal, collisions with collision time τ have a damping effect in the equation of motion for the momentum of an electron. This effect is:
   a. proportional to $-\tau$
   b. proportional to $-1/\tau$
   c. proportional to τ^2
   d. proportional to $-1/\tau^2$

2. The Drude model was used to explain the Wiedemann-Franz law. The Wiedemann-Franz law states that the thermal to electrical conductivity ratio (κ / σ) is
   a. proportional to T
   b. proportional to 1/T
   c. proportional to T^2
   d. proportional to $1/T^2$

3. From the Drude model, one can show that for certain frequencies of electromagnetic radiation, a metal can be "transparent." If f_p is the plasma frequency in Hz, transparency will occur for:
   a. $f < f_p$ and $\varepsilon < 0$
   b. $f > f_p$ and $\varepsilon < 0$
   c. $f < f_p$ and $\varepsilon > 0$
   d. $f > f_p$ and $\varepsilon > 0$

4. Compared to Drude theory, Sommerfeld finds for the heat capacity c_v and mean square velocity $<v^2>$ at room temperature, that:
   a. c_v is ~ $c_{v,classical}$ / 100 and $<v^2> \sim <v^2>_{classical}$ /100
   b. c_v is ~ $c_{v,classical}$ x 100 and $<v^2> \sim <v^2>_{classical}$ /100
   c. c_v is ~ $c_{v,classical}$ / 100 and $<v^2> \sim <v^2>_{classical}$ x100
   d. c_v is ~ $c_{v,classical}$ x 100 and $<v^2> \sim <v^2>_{classical}$ x100

5. What is the condition for the validity of the semiclassical Sommerfeld theory?
   a. the electron's position x must be known to within less than a few Å
   b. the uncertainty Δx of electron's position must be much larger than a few Å
   c. that momentum p of electron must be small compared to the Fermi momentum p_F
   d. than uncertainty Δp of electron's momentum must be much greater than p_F

6. In three dimensions, the density of levels $g(\varepsilon)$ is proportional to:
    a. energy (ε)
    b. energy squared (ε^2)
    c. inverse of energy (ε^{-1})
    d. square root of energy ($\sqrt{\varepsilon}$)

7. Any crystal lattice can be described as:
    a. a (2-dimensional) "Bravais" lattice (net) with a basis
    b. a (3-dimensional) Bravais lattice
    c. a (3-dimensional) Bravais lattice with a basis
    d. a simple cubic lattice with a basis
    e. a hexagonal close-packed lattice with a basis

8. A Wigner-Seitz primitive cell:
    a. contains only 1 lattice point but lacks the symmetry of its Bravais lattice
    b. when translated through all lattice vectors, will *not* fill up all the space
    c. contains only 1 lattice point and has the symmetry of its Bravais lattice
    d. depends on the specific choice of primitive lattice vectors

9. Which of the following lattices is *not* a Bravais lattice?
    a. monoclinic
    b. tetragonal
    c. body-centered cubic
    d. hexagonal close-packed
    e. orthorhombic

Part II. Multi-Part Problems. (**20 pts each**)

Problem #1: Derive the AC Electrical conductivity of a metal.

a. (7 pts) Assume the following equation for momentum of an electron in the Drude model

$$\frac{d\mathbf{p}}{dt} = \frac{-\mathbf{p}}{\tau} + \mathbf{f}$$

Assume that electric field **E**(t), momentum **p**(t), and current density **j**(t) are all oscillating functions of time, such as:

$$\mathbf{E}(t) = \mathrm{Re}[\ \mathbf{E}(\omega)\, e^{-i\omega t}\ ]$$

Insert the time-varying quantities into momentum equation and obtain a simple equation relating **p**(ω) to **E**(ω).

b. (7 pts) Write down the simple formula relating current density **j** to momentum **p** involving n, e, and m. (Hint: what is **j** in terms of **v**?) Using this formula, find an equation relating **j**(ω) to **E**(ω).

c. (6 pts) What is the frequency-dependent proportionality factor between **j** and **E**? (i.e. **j** = ?? **E**) Extract this factor from the result of part (b), rearrange it, and write the final result for this factor in terms of the DC electrical conductivity σ_o, frequency ω, and collision time τ.

(a)

$$\vec{p}(t) = \mathrm{Re}\left[\vec{p}(\omega)\, e^{-i\omega t}\right] \qquad \vec{j}(t) = \mathrm{Re}\left[\vec{j}(\omega)\, e^{-i\omega t}\right]$$

$$\mathrm{Re}\left[-i\omega\,\vec{p}(\omega)\, e^{-i\omega t} = -\frac{\vec{p}(\omega)}{\tau}\, e^{-i\omega t} - e\,\vec{E}(\omega)\, e^{-i\omega t}\right]$$

$$\Rightarrow \quad \left(-i\omega + \frac{1}{\tau}\right)\vec{p}(\omega) = -e\,\vec{E}(\omega)$$

$$\Rightarrow \quad \boxed{\vec{p}(\omega) = \frac{e}{\left(i\omega - \frac{1}{\tau}\right)}\,\vec{E}(\omega)}$$

(b) $$\vec{j}(\omega) = -ne\frac{\vec{p}}{m}$$

$$\boxed{\vec{j}(\omega) = \frac{ne}{m}\left(\frac{e}{1/\tau - i\omega}\right)\vec{E}(\omega)}$$

(c) $$\vec{j}(\omega) = \frac{ne^2\tau}{m}\frac{1}{(1-i\omega\tau)}\vec{E}(\omega)$$

$$\boxed{\vec{j}(\omega) = \frac{\sigma_0}{(1-i\omega\tau)}\vec{E}(\omega) \quad \text{w\textbackslash} \quad \sigma_0 = \frac{ne^2\tau}{m}}$$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}}$$

Problem#2: Estimate the specific heat of a metal in the Sommerfeld model.

a. (3 pts) Draw a picture of the Fermi function at temperature T > 0 in the vicinity of the Fermi level ε_F. Compare to the picture at T = 0.

b. (3 pts) Explain in a few sentences, what the shape of this function means in terms of excitation of electrons at T > 0 compared to the situation at T = 0.

c. (3 pts) Assume a density of states $g(\varepsilon)$. Estimate the number density of electrons which are excited at temperature T > 0, in terms of k_B, T, and $g(\varepsilon)$.

d. (4 pts) What is the excitation energy per electron? Using that and the result from (c), estimate the increase in energy density (Δu) due to the temperature excitation.

e. (3 pts) What is the constant volume specific heat c_v in terms of u and T? (hint: involves a derivative). Use that expression to determine c_v.

f. (4 pts) Compare your result to the classical value of constant-volume specific heat

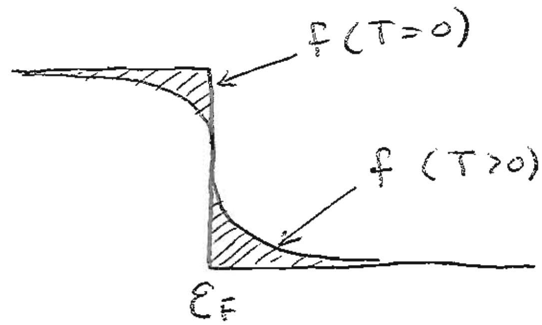
$$(3/2)nk_B$$

For the comparison, make use of the equation which states that:

$$g(\varepsilon_F) = (3/2)n/\varepsilon_F$$

Is the Sommerfeld c_v larger or smaller than the classical one? By what factor?

(a)



(b) electrons below ε_F are excited into states above ε_F, at $T > 0$

(c) $n_{ex} \simeq \Delta\varepsilon \times$ states/unit energy $(g(\varepsilon))$

$$\simeq (k_B T)\, g(\varepsilon_F)$$

$$\boxed{n_{ex} = k_B T\, g(\varepsilon_F)}$$

(d) excitation energy per electron

$$\boxed{\mathcal{E}_{ex} \simeq k_B T}$$

$$\Rightarrow \quad \Delta u \simeq \mathcal{E}_{ex} \times n_{ex}$$

$$= k_B T \times k_B T \, g(\mathcal{E}_F)$$

$$\boxed{\Delta u = (k_B T)^2 g(\mathcal{E}_F)}$$

e. $$C_v = \left(\frac{\partial u}{\partial T}\right)_n = \boxed{2\, g(\mathcal{E}_F)\, k_B^2 T}$$

f. $$C_{v,\,class} = \frac{3}{2} n k_B \qquad \text{use} \quad g(\mathcal{E}_F) = \frac{3}{2}\frac{n}{\mathcal{E}_F}$$

$$\rightarrow C_v = 2\left(\frac{3}{2}\frac{n}{\mathcal{E}_F}\right) k_B^2 T$$

$$C_v = 3\left(\frac{k_B T}{\mathcal{E}_F}\right) n k_B = \frac{3}{2}\left(2\frac{k_B T}{\mathcal{E}_F}\right) n k_B$$

$\Rightarrow$ C_v is ~~[illegible]~~ smaller than $C_{v,\,classical}$ by factor $\dfrac{2 k_B T}{\mathcal{E}_F} \sim$ ~~[illegible]~~ 10^{-2}

$\sim .01$

Problem#3: Properties of *face-centered cubic* lattice.

a. (4 pts) Consider a face-centered cubic lattice. Is the lattice a Bravais lattice? Are the "corner" sites and the "face-centered" sites *inequivalent* or *equivalent*? Draw a picture of the fcc lattice in 3-D.

b. (4 pts) Write down a "symmetric" set of primitive lattice vectors for fcc. Do these primitive lattice vectors "span" the fcc lattice?

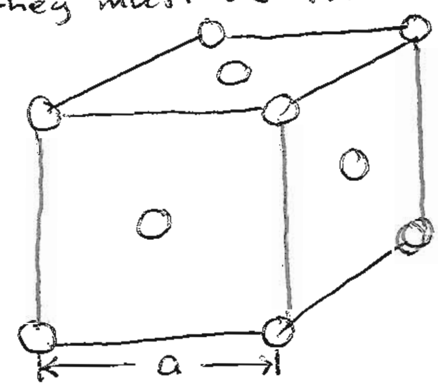
c. (4 pts) Consider that the length of the fcc conventional cube side is a. What is the volume of a *primitive cell* of the fcc lattice in terms of a? Also, what is the density of lattice points (number per unit volume) in terms of a?

d. (4 pts) Consider a spherical "atom" placed on each lattice site. Assume that these spheres are "close-packed". How many such close-packed spheres are contained within each *primitive cell*? What is the radius of such a close-packed sphere in terms of a?

e. (4 pts) What is then the volume of one of the close-packed spheres? Based on this and the answer from part (c), calculate the *packing fraction* for the close-packed spheres in the fcc lattice.

(a) Yes, fcc is Bravais.

corner and face-centered sites are equivalent (they must be for Bravais)



(b) $\vec{a}_1 = \frac{a}{2}(\hat{y}+\hat{z}) \quad \vec{a}_2 = \frac{a}{2}(\hat{z}+\hat{x}) \quad \vec{a}_3 = \frac{a}{2}(\hat{x}+\hat{y})$

yes, these span the fcc lattice

(c) volume of primitive cell

for fcc, there are 4 lattice sites/conventional c

there should be 1 lattice site in each primitive cell

$\Rightarrow$ primitive cell volume = $\dfrac{\text{conventional cell volume}}{4}$

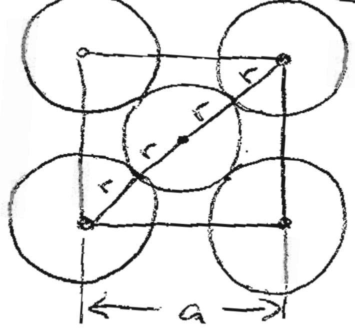
$$\Rightarrow \boxed{V_{prim} = \frac{a^3}{4}}$$

density of lattice points = $\dfrac{1}{V_{prim}}$

$$\boxed{n = \frac{4}{a^3}}$$

(d) each primitive cell contains 1 lattice site

$\Rightarrow$ contains 1 "spherical atom" or, 1 close-packed sphere



$$\Rightarrow \boxed{r_{sph} = \frac{\sqrt{2}\,a}{4}}$$

(e) $V_{sphere} = \dfrac{4}{3}\pi r_{sph}^3 = \dfrac{4}{3}\pi \dfrac{2\sqrt{2}}{4^3} a^3 = \dfrac{2\sqrt{2}\,\pi}{3\times 16} a^3 = \dfrac{\sqrt{2}\,\pi}{24} a^3$

close-packed packing fraction for fcc = $\dfrac{V_{sph}}{V_{prim}} = \dfrac{\frac{\sqrt{2}\,\pi}{24} a^3}{\frac{1}{4} a^3} = \boxed{\dfrac{\sqrt{2}\,\pi}{6}}$

$= 0.74$