Physics 615, Fall 2003, Final Exam

Time: Two hours

Answer all questions. Partial credit for incomplete answers will be given. **Explain your answers in detail.**

1. (35 pts.) Using contour integration, or otherwise, compute the integrals (m is a positive integer):

$$\text{(a)} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}, \qquad \text{(b)} \int_{-\pi}^{\pi} (\cos x)^{2m}\, dx.$$

2. (35 pts.) Consider the hypergeometric differential equation (with $c > 1$):

$$x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0. \tag{1}$$

   (a) What kind of point is $x = 0$ for this differential equation?

   (b) Propose a series solution around $x = 0$. Obtain the indicial equation $P(k) = 0$ and show that one of its solutions is $k = 0$. From now on (except at the last point), we focus on the solution for $k = 0$.

   (c) Obtain a series solution y(x) of the equation, such that $y(x = 0) = 1$. Write the terms up to order x^3 explicitly.

   (d) Write the generic term for the series.

   (e) Find the radius of convergence of the series.

   (f) Assuming that c is not an integer, what is the leading order behavior of the second solution, for x near zero?

3. (30 pts.) Consider the diffusion equation for the particle density $\rho(x,t)$:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{2}$$

   with the boundary conditions: $\rho(x = 0, t) = 0 = \rho(x = L, t)$.

   (a) Write the most general solution of the differential equation that is consistent with the boundary conditions at $x = 0$ and $x = L$.

   (b) Assume that the initial condition is $\rho(x, t = 0) = f(x)$. Find $\rho(x,t)$ for all x, t.

   (c) Assume that the initial condition is $\rho(x, t = 0) = \delta(x - x_0)$. Find $\rho(x,t)$ for all x, t.

   (d) For the case when the initial condition is a delta function, draw a sketch of your solution showing how it changes in time.

Some (possibly useful) formulas:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \qquad \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1), \qquad \int \frac{dx}{x} = \ln|x| + C, \qquad \int e^x dx = e^x + C$$

$$\int \cos(x)dx = \sin(x) + C, \qquad \int \sin(x)dx = -\cos(x) + C$$

$$\int_{-\infty}^{\infty} dx\ \delta(x-a)f(x) = f(a), \qquad \delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$x > y \Rightarrow e^x > e^y, \qquad x > y > 0 \Rightarrow \ln x > \ln y$$

$$\int_{-\infty}^{\infty} dx\ \delta(x-a)f(x) = f(a), \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikx}, \qquad \int_{-\infty}^{\infty} dx\ e^{-\frac{\alpha}{2}x^2 + i\beta x} = \sqrt{\frac{2\pi}{\alpha}}\ e^{-\beta^2/2\alpha}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\tilde{f}(k)$$

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

$$(1+z)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^j$$