Chapter 2

More on the Nucleon-Nucleon Force

2.1 Meson Exchange in the Era of QCD

It should be of no surprise that the study of nuclear forces has attracted the interest of many theoretical as well as experimental physicists and has been the topic of vigorous research for more than fifty years. It not only represents the force between two nucleons and the basis for an understanding of nuclei, but it is also a prominent example of a strong interaction between particles.

Nowadays, almost everybody believes that quantum chromodynamics (QCD)is the theory of strong interactions. Therefore, the nucleon-nucleon (NN) interaction is completely determined by the underlying dynamics of the fundamental constituents of hadrons, i.e., quarks and gluons. Nevertheless, the meson exchange picture keeps its validity as a suitable effective description of the NN interaction in the low-energy region relevant in nuclear physics for the following reasons: Due to asymptotic freedom, QCD in terms of quarks and gluons can be treated perturbatively only for large momentum transfers, i.e., at distances smaller than 0.2 fm or so, which typically occur in high-energy physics processes. At larger distances relevant in nuclear physics, a description of strong interaction processes in terms of the fundamental constituents, because of the highly non-perturbative character of QCD in this region, becomes extremely complicated and is not possible, neither at present no in the foreseeable future.

Fortunately, at distances larger than the nucleon extension, which dominate nuclear physics phenomena, color confinement dictates that nucleons can only interact by exchanging colorless objects, i.e., just mesons. Only at smaller distances, at which the two nucleons overlap, genuinely new processes may occur involving explicit quark-gluon exchange. Due to the repulsive core of the NN interaction, however, both nucleons do not come very close to each other unless the scattering energy is rather high. Thus, there is good reason to believe these processes not to dominate the NN interaction for energies relevant in nuclear physics. Consequently, meson exchange (to be considered as a convenient, effective description of complicated quark-gluon processes) should remain a valid concept for deriving a realistic NN interaction, representing a reliable starting point for nuclear structure calculations.

2.2 What Do We Know Empirically About the Nuclear Force?

The basic qualitative features of the nuclear force are the following [1]:

a) Nuclear forces have a finite range, in contract to the Coulomb force. This can be easily deduced from the saturation properties of heavy nuclei: Here, the binding energy per nucleon as well as the density are nearly constant. If the nuclear force were of long range, both quantities would increase with the nucleon number.

b) The nuclear force is attractive at intermediate ranges. The attractive character of the nuclear force is clearly established in nuclear binding. The range of this attraction can be obtained from the central density of heavy nuclei, which is about 0.17 fm^{-3}, giving to each nucleon a volume of about 6 fm^3. Therefore, in the interior of a heavy nucleus, the average distance between two nucleons is roughly 2 fm.

c) The nuclear force is repulsive at short distances. This is most easily seen in the empirical ${}^1S_0$ and ${}^1D_2$ NN partial wave phase shifts (in the conventional notation ${}^{2S+1}L_J$, where $L(J)$ denotes the orbital (total) angular momentum and S the total spin), which are deduced from NN scattering data (cross sections, polarization observables) by means of a phase shift analysis. For small lab energies (up to about 250 MeV), the ${}^1S_0$ phase shift is positive, which corresponds to attraction. For high energies, it becomes negative (equivalent to repulsion), whereas the ${}^1D_2$ phase shift stays positive up to about 800 MeV. This is consistent with a repulsion of short range since an S-state is sensitive to the inner part of the force, whereas in a D-state the nucleons are kept apart by the centrifugal barrier.

d) The nuclear force contains a tensor part. This is most clearly established in the presence of a deuteron quadrupole moment and the so-called D/S ratio of the deuteron wave function [2].

e) The nuclear force contains a spin-orbit part. This is clearly seen in nuclear spectra. Furthermore, a quantitative description of triplet-P waves require a strong spin-orbit force.

There are additional spin-dependent terms in the NN force, namely a spin-spin and a quadratic spin-orbit term. They are, however, of minor importance.

In fact, from general invariance principles (translation, Galilei, rotation, parity, time-reversal), the most general form of a nonrelativistic potential contains just these five terms, i.e., central (c), spin-spin(s), tensor (t), spin-orbit (LS) and quadratic spin-orbit (LL):

$$V = \sum_i V_i \, O_i \tag{2.1}$$

with

$$\begin{aligned} O_c &= \mathbf{1} \\ O_s &= \vec{\sigma}_1 \cdot \vec{\sigma}_2 \\ O_t &\equiv S_{12} \equiv \frac{3\vec{\sigma} \cdot \vec{r}\,\vec{\sigma}_2 \cdot \vec{r}}{r^2} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \\ O_{LS} &= \vec{L} \cdot \vec{S} \\ O_{LL} &= (\vec{L} \cdot \vec{S})^2 \end{aligned} \tag{2.2}$$

where $\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$.

The coefficients V_i can in general depend on the distance r, the relative momentum $\vec{p}^{\,2}$ and $\vec{L}^2$; they are completely undetermined. Phenomenological potentials like the Hamada-Johnson [3], the Reid [4] potential or its update by the Nijmegen group make an ansatz for $V_i(r)$, with a total of about 50 parameters, and fix these by adjusting them to the NN scattering data. However, such potentials cannot provide any basic understanding of the interaction mechanism, and the parameters have no physical meaning.

2.3 Historical Background

The development of a microscopic theory of nuclear forces started around 1935 with Yukawa's fundamental hypothesis [5] that the nuclear force is generated by massive-particle exchange, leading to an interaction of the type $\frac{e^{-m_\alpha r}}{r}$ where m_α is the mass of the exchange particle and r is the distance between two nucleons. This is quite analogous to the electromagnetic case in which the interaction is known to be generated by (massless) photon exchange yielding the well-known Coulomb-potential being proportional to $\frac{1}{r}$.

The original Yukawa idea of a scalar field interacting with nucleons was soon extended to vectors (Proca [6]) and to pseudoscalar and pseudovector fields (Kemmer [7]). The consideration of a pseudoscalar field was dictated by the discovery of the quadrupole moment of the deuteron [8], whose sign was correctly given by the exchange of an (isovector) pseudoscalar meson. Almost ten years later, in 1947, a pseudoscalar meson, the pion, was indeed found [9].

The next period started around 1950, and again Japanese physicists initiated it. Taketani, Nakamura and Sasaki [10] (TNS) proposed to subdivide the range of the nuclear force into three regions: a "classical" (long range, $r > 2\ fm$), a "dynamical" (intermediate range, $1\ fm < r < 2\ fm$) and a "core" (short range, $r \leq 1\ fm$) region. The classical region is dominated by one-pion exchange (OPE). In the intermediate range, the two-pion exchange (TPE) is supposed to dominate, although heavier-meson exchange (to be introduced later) become relevant, too. Finally, in the core region, many different processes should play a role: multi-pion, heavy-meson and (according to our current understanding) genuine quark-gluon exchange. Thus, in view of QCD-inspired approaches to the nuclear force, this division is still most meaningful.

In the 1950's, the one-pion exchange became well established as the long-range part of the nuclear force. Tremendous problems occurred, however, when the 2π exchange contribution to the NN interaction was attacked. Apart from various uncertainties in the results (the best known being those of Taketani, Machida and Onuma [11], and Brueckner and Watson [12]), it was impossible to derive a sufficient spin-orbit force from the 2π exchange [13]. For that reason, Breit [14] in 1960 suggested to look for heavy vector bosons in order to account for the empirically well-established, short-ranged spin-orbit force. In fact, such mesons (ρ, ω) with a mass of nearly 800 MeV were soon discovered [15].

This led to the next step, namely the development of one-boson exchange (OBE) models. Their basic assumption is that multi-pion exchange can be well accounted for by the exchange of multi-pion resonances, i.e., that uncorrelated multi-pion exchange (apart from iterative contributions which are generated by the unitarizing equation) can be neglected. Such OBE models [16, 17, 18] (the contribution of the Bonn group is reviewed in Ref.. [17]), provide a relatively simple expression for the nuclear force; indeed, they can account quantitatively for the empirical NN data using only very few parameters and thus convincingly demonstrate the importance of correlated interactions with two (or more) pions.

In all OBE models, the intermediate-range attraction is generated by the exchange of a scalar-isoscalar boson with a mass around 600 MeV (representing a $2\pi S$-wave resonance), which, although appearing in Particle Data Tables of the sixties, has not been confirmed empirically. This has to be considered as a serious drawback of OBE models. Therefore, the program of a realistic 2π-exchange calculation was taken up again; however, in contract to the fifties, with the goal to include not only the uncorrelated 2π-exchange contribution involving nucleon intermediate states, but also those involving nucleon excitations like the Δ isobar. Furthermore, from the experience with OBE models, it was clear from the beginning that correlated 2π exchanges should be included, too.

In the dispersion-theoretic approach to the 2π exchange, empirical πN- (and $\pi\pi-$) data are used to derive the corresponding NN amplitude, with the help of causality, unitarity and crossing. Correlated as well as uncorrelated 2π exchange is automatically included.

Corresponding NN potentials are developed in the 1970's, in particular by the Stony Brook [19] and the Paris [20] group, adding to the dispersion-theoretic 2π-exchange contribution OPE- and ω-exchange as well as some arbitrary phenomenological potential of essentially short-range nature. In case of the Paris-potential, the final result is parameterized by means of static Yukawa terms [21].

However, such a simplified representation of the nuclear force is probably insufficient in many areas of nuclear physics. For example, a consistent evaluation of three-body forces and meson-exchange current corrections to the electromagnetic properties of nuclei requires an explicit and consistent description of the NN interaction in terms of field-theoretic vertices. Also, a well-defined off-shell behavior and modifications of the nuclear force when inserted into the many-body problem (e.g., Pauli-blocking of the 2π-exchange contribution) are natural consequences of meson exchange. Only a field-theoretical approach can account for these.

Work along the field theoretical line was taken up in the late 1960's by Lomon and collaborators [22, 23]. They evaluated the 2π-exchange Feynman diagrams with nucleons and represented their result in the framework of the relativistic three-dimensional reduction of the (four-dimensional) Bethe-Salpeter [24] equation, suggested by Blankenbecler and Sugar [25]. In subsequent work [23], they also studied the correlated $2\pi S$-wave contribution. However, they did not include processes involving the Δ-isobar (an excited state of the nucleon with a mass of 1232 MeV and spin-isospin 3/2) in intermediate states, which are known to contribute substantially to the nuclear force. Further, nonresonant 3π- and 4π-exchange has to be considered since their range is about that of ω-exchange, which is included in all models.

Since the 1970's the Bonn group has pursued a program that includes all relevant diagrams in a field-theoretical model. In the early period [26], a relativistic three-dimensional equation was used together with the principle of minimal relativity [27]; later the treatment has been based on relativistic, time-ordered perturbation theory [28]. A final status of the Bonn model is described in detail in Ref. [29].

Let me finally mention attempts to derive the nucleon-nucleon interaction in the constituent quark model, based on one-gluon exchange. Corresponding calculations started about 15 years ago and many groups have been involved [30]. Indeed, certain qualitative features of the short-range part of the NN interaction emerged, namely some inner repulsion and spin-orbit force. However, all models of this kind create either too little or no intermediate-range attraction (which is sometimes artificially cured by adding a suitable attraction arising from scalar boson exchange). Note that pion exchange has to be added in any case. Thus, the meson exchange concept for constructing the NN interaction clearly keeps its validity at the low and intermediate energies relevant to nuclear physics.

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2.4 Bethe-Salpeter Equation

In nonrelativistic scattering the equation describing the scattering process is the Lippmann-Schwinger equation. In a relativistic description, one starts from the Bethe-Salpeter equation (derived $\approx$ 1950), in which the relativistic scattering amplitude M is given by processes like

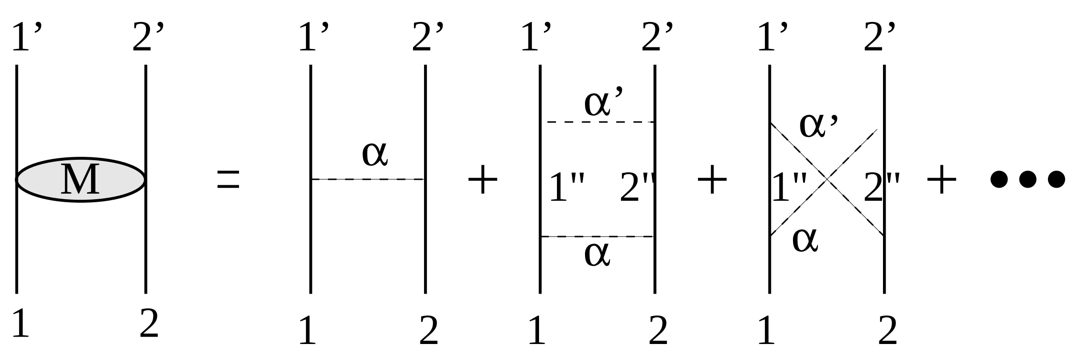

Figure 2.1.1 Relativistic scattering amplitude M

Analogously to the Lippmann-Schwinger equation, the Bethe-Salpeter equation can be written as integral equation

$$\begin{aligned} M &= K + KGM \\ &= K + KGK + KGKGK + \cdots \end{aligned} \tag{2.3}$$

where K is the sum of all irreducible diagrams.

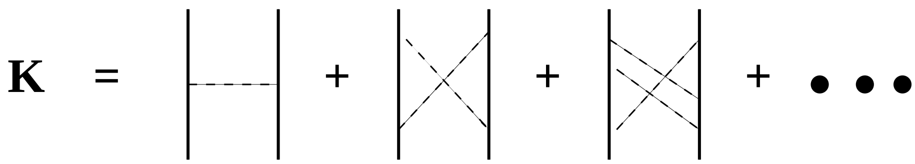

Figure 2.1.2: Irreducible kernel of the BS equation

K can be interpreted as a "relativistic potential." Take, e.g., the first term in Fig. 2.1.2 as "potential," i.e.,

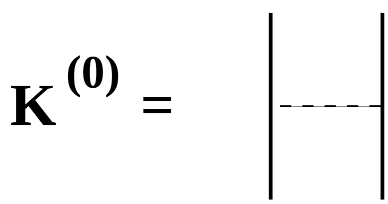

Figure 2.1.3: Largest order diagram to K

Then the scattering amplitude $M^{(0)}$ takes the following form:

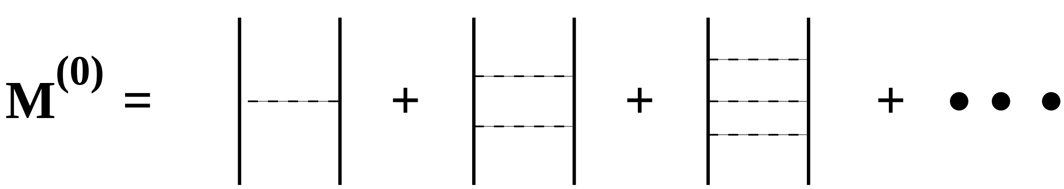

Figure 2.1.4: Interaction of $K^{(0)}$

$$M^{(0)} = K^{(0)} + K^{(0)} G K^{(0)} + K^{(0)} G K^{(0)} G K^{(0)} + \cdots \tag{2.4}$$

Eq. (2.4) represents the so-called "ladder approximation" of the BS equation. As it turns out, it is **not** a good approximation to the BS equation, since the crossed diagrams are equally important as part of the irreducible kernel.

G is the two-nucleon propagator and describes the two nucleons in the intermediate states. Nucleons are spin-$\frac{1}{2}$ particles, thus a starting point for obtaining the propagator is the Dirac equation. Define $S^{\alpha,\beta}(x,y)$ with $\alpha, \beta = 1,2,3,4$ as propagator (4×4 matrix). Insertion into the Dirac equation leads to

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\mathbf{1}\right)_{\alpha\gamma} S_{\gamma\beta}(x,y) = \delta^{(4)}(x-y)\,\delta_{\alpha\beta} \tag{2.5}$$

Ansatz:

$$S_{\gamma\beta}(x,y) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq(x-y)}\, S_{\gamma\beta}(q) \tag{2.6}$$

which gives with (2.5)

$$\begin{aligned} \int \frac{d^4q}{(2\pi)^4} (\gamma^\mu q_\mu - m\mathbf{1})_{\alpha\gamma}\, S_{\gamma\beta}(q)\, e^{-iq(x-y)} &= \delta^{(4)}(x-y)\,\delta_{\alpha\beta} \\ &= \int \frac{d^4q}{(2pi)^4}\, e^{-iq(x-y)}\, \delta_{\alpha\beta} \end{aligned} \tag{2.7}$$

In the last relation, the definition of the δ-function was used. A comparison of the coefficients gives

$$(\gamma^\mu q_\mu - m\mathbf{1})_{\alpha\gamma}\, S_{\gamma\beta}(q) = \delta_{\alpha\beta} \tag{2.8}$$

This leads to the Ansatz for the two-nucleon propagator in momentum space

$$S_{\gamma\beta}(q)\, \frac{(\gamma^\mu q_\mu + m\mathbf{1})_{\gamma\beta}}{q^2 - m^2}\,, \tag{2.9}$$

where $q^2 = q^\mu q_\mu$. Proof by insertion into (2.8):

$$\begin{aligned}(\gamma^\mu q_\mu - m\mathbf{1})_{\alpha\gamma}\,(\gamma^\nu q_\nu + m\mathbf{1})_{\gamma\beta} &= (\gamma^\mu\gamma^\nu\, q_\mu q_\nu - m^2\mathbf{1})_{\alpha\beta} \\ &= \left(\frac{1}{2}\,[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]\, q_\mu q_\nu - m^2\mathbf{1}\right)_{\alpha\beta} \\ &= (G^{\mu\nu}\mathbf{1}\, q_\mu q_\nu - m^2\mathbf{1})_{\alpha\beta} \\ &= (q^\nu q_\nu - m^2)\,(\mathbf{1})_{\alpha\beta} \\ &= (q^2 - m^2)\,\delta_{\alpha\beta}\end{aligned}$$

which shows that the Ansatz for $S_{\alpha\beta}$ was correct. Thus, the two nucleon propagation is given as

$$S(q) = \frac{\gamma^\nu q_\nu + m\mathbf{1}}{q^2 - m^2 + i\epsilon} =: \frac{1}{\gamma^\nu q_\nu - m\mathbf{1} + i\epsilon} \tag{2.10}$$

Since there are singularities at $q^2 = m^2$, $S(q)$ has to be defined off-mass shell. Thus, one obtains for the two-nucleon propagator:

$$\begin{aligned}G &= \left[\frac{1}{\gamma^\mu q_\mu^{(1)} - m\mathbf{1} + i\epsilon}\right]^{(1)} \left[\frac{1}{\gamma^\mu q_\mu^{(2)} - m\mathbf{1} + i\epsilon}\right]^{(2)} \\ &= \left[\frac{\gamma^\mu q_\mu^{(1)} + m\mathbf{1}}{q^{(1)\,2} - m^2 + i\epsilon}\right]^{(1)} \left[\frac{\gamma^\mu q_\mu^{(2)} + m\mathbf{1}}{q^{(2)\,2} - m^2 + i\epsilon}\right]^{(2)}\end{aligned} \tag{2.11}$$

2.5 Relativistic Kinematics

2.5.1 Single Scattering

Consider the following diagram.

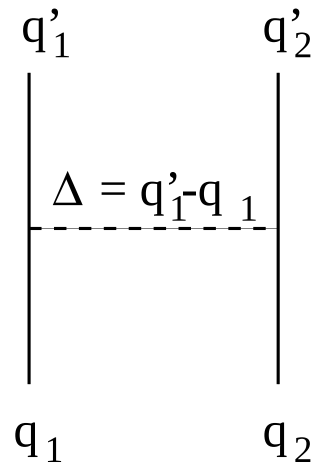

Figure 2.2.1: Single scattering diagram

with

$$
\begin{array}{rclcrcl}
q_1^\mu & = & (q_1^0, \vec{q}_1) & ; & q_1^{\mu\prime} & = & (q_1^{0\prime},\ \vec{q}\,'_1) \\
q_2^\mu & = & (q_2^0, \vec{q}_2) & ; & q_2^{\mu\prime} & = & (q_2^{0\prime}, \vec{q}_2\,')
\end{array}
\qquad (2.12)
$$

and let Δ be an exchanged meson, e.g., at the vertices, one has 4-momentum conservation, i.e.,

$$\Delta \;=\; q_1' \;-\; q_1 \;=\; q_2 \;-\; q_2' \qquad (2.13)$$

and conservation of the total momentum P:

$$P' \;=\; q_1' \;+\; q_2' \;=\; q_1 \;+\; q_2 \;=\; P \qquad (2.14)$$

We define center-of-mass momentum P^μ and relative momentum q^μ as

$$
\begin{array}{rcl}
P^\mu & = & (p^0, \vec{p}) \;=\; q_1^\mu \;+\; q_2^\mu \\
q^\mu & = & (q^0, \vec{q}) \;=\; \frac{1}{2}\,(q_1^\mu \;-\; q_2^\mu) \\
q^{\mu'} & = & (q^{0\prime}, \vec{q}\,') \;=\; \frac{1}{2}\,(q +_1^{\mu\prime} \;-\; q_2^{\mu\prime})
\end{array}
\qquad (2.15)
$$

From (2.14) and (2.15) follows

$$q_1^\mu \;=\; \frac{1}{2}\,P^\mu \;+\; q^\mu$$

$$
\begin{aligned}
q_2^\mu &= \frac{1}{2}\,P^\mu \;-\; q^\mu \\
q_1^{\mu'} &= \frac{1}{2}\,P^\mu \;+\; q^{\mu'} \\
q_2^{\mu'} &= \frac{1}{2}\,P^\mu \;-\; q^{\mu'}
\end{aligned}
\tag{2.16}
$$

and thus $q_1^{\mu'} \;-\; q_1^{\mu'} \;=\; q^{\mu'} \;-\; q^\mu$. In the c.m. system, we have $\vec{P} \;=\; 0$, thus

$$
\begin{aligned}
q_1^\mu &= \left(\frac{1}{2}\,P^0 \;+\; q^0, \vec{q}\right) \\
q_2^\mu &= \left(\frac{1}{2}\,P^0 \;-\; q^0, \;-\;\vec{q}\right) \\
q_1^{\mu'} &= \left(\frac{1}{2}\,P^0 \;+\; q^{0'}, \vec{q}\,'\right) \\
q_2^{\mu'} &= \left(\frac{1}{2}\,P^0 \;-\; q_1', \;-\;\vec{q}\,'\right)
\end{aligned}
\tag{2.17}
$$

and for the exchange particle

$$
\Delta \;=\; (q^{0\prime} \;-\; q^0,\; \vec{q}\,' \;-\; \vec{q}) \;=\; (q^{\mu'} \;-\; q^\mu)
\tag{2.18}
$$

In the initial state, particles 1 and 2 are on-mass-shell (real particles), thus

$$
\begin{aligned}
(q_1^\mu)^2 &= m^2 \;=\; (q_1^0)^2 \;-\; (\vec{q}_1)^2 \\
q_1^0 &= \sqrt{q_1^2 \;+\; m^2} \;:=\; E_{q_1} \;=\; E_q \\
q_2^0 &= \sqrt{q_2^2 \;+\; m^2} \;:=\; E_{q_2} \;=\; E_q
\end{aligned}
\tag{2.19}
$$

and

$$
\begin{aligned}
P^0 &= q_1^0 \;+\; q_2^0 \;=\; 2E_q \\
q^0 &= \frac{1}{2}\,(q_1^0 \;-\; q_2^0) \;=\; 0
\end{aligned}
\tag{2.20}
$$

Thus (suppressing the 4-indices):

$$
\begin{aligned}
q_1 \;=\; (E_q, \vec{q}) \quad &; \quad q_2 \;=\; (E_q, \;-\;\vec{q}) \\
q_1' \;=\; (E_q \;+\; q^{0'},\; \vec{q}\,') \quad &; \quad q_2' \;=\; (E_q \;-\; q^{\prime,0},\; -\;\vec{q}\,')
\end{aligned}
\tag{2.21}
$$

For the single scattering diagram in Figure 2.2.1, particles 1 and 2 are also on-mass-shell, i.e., $q_1^{0\prime} = q_2^{0\prime} = \sqrt{q^2 + m^2} := E_{q'}$, from which follows

$$\begin{aligned} P^0 &= 2E_{q'} \\ q^{0'} &= 0 \\ q_1' &= (E_{q'},\ \vec{q}\,') = (E_q,\ \vec{q}\,') \\ q_2' &= (E_{q'}\ -\ \vec{q}\,') = (E_q,\ -\ \vec{q}') \end{aligned} \tag{2.22}$$

from which follows $|\ \vec{q}\,'\ | = |\ \vec{q}\ |$.

Thus, for the physical scattering, the single scattering diagram looks as follows:

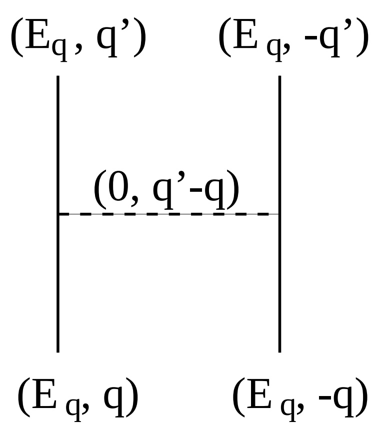

Figure 2.2.2: Kinematics of the single scattering

2.5.2 Double Scattering

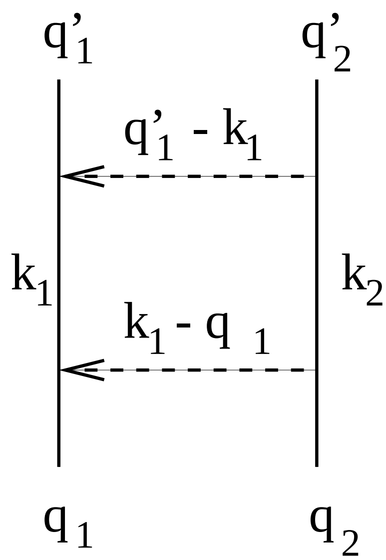

Figure 2.2.3: Double scattering diagram

Here K_1, K_2 describe intermediate states. Again one has 4-momentum conservation at **all** vertices, i.e.,

$$P = q_1 + q_2 = k_1 + k_2 = q_1' + q_2' \tag{2.23}$$

and

$$K = \frac{1}{2}(k_1 - k_2) . \tag{2.24}$$

Let us consider the c.m. system. Again, in the initial state particles 1 and 2 are on-mass shell, and in the final state both particles are on-mass shell. However, in the intermediate states, the particles can have all possible **independent** values of $k^0, \vec{k}$, i.e., both particles are in general off-mass shell in the intermediate states, i.e., are virtual states.

Thus, one has the following kinematic diagram:

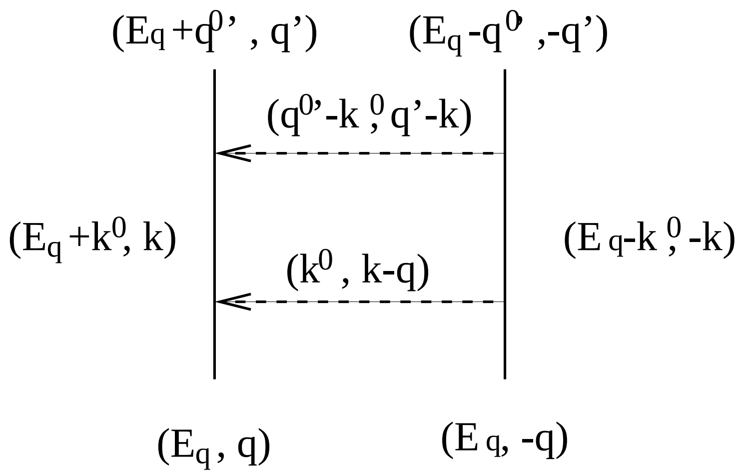

Figure 2.2.4: Kinematics of the double scattering

In the c.m. system, the Bethe-Salpeter equation can be written as ($\vec{P} \; = \; 0$)

$$M(q',q \mid P^0) \; = \mathbf{K}(q',q \mid P^0) \; + \; \frac{1}{(2\pi)^4} \int d^4k \; \mathbf{K}(q',k \mid P^0) \; G(k,P^0) \; M(k,q \mid P^0) \; . \tag{2.25}$$

Here $P^0 \; = \; 2E_q$, i.e., in the initial state both particles are on-mass shell. The integration implies all possible intermediate states. The propagator $G(k,P^0)$ is given as

$$\begin{aligned} G(k,P^0) \quad &= \quad \left[\frac{1}{\gamma^\nu \; (\frac{1}{2} \; P + k)_\nu \; - \; m\mathbf{1} \; + \; i\epsilon}\right]^{(1)} \left[\frac{1}{\gamma^\nu \; (\frac{1}{2} \; P \; - \; k)_\nu \; - \; m\mathbf{1} \; + \; i\epsilon}\right]^{(2)} \\ &\overset{(c.m.)}{=} \left[\frac{1}{\gamma^0 \; (\frac{1}{2} \; P^0 + k^0) - \; \vec{\gamma} \; \cdot \; \vec{k} - \; m\mathbf{1} \; + \; i\epsilon}\right]^{(1)} \\ &\times \quad \left[\frac{1}{\gamma^0 \; (\frac{1}{2} \; P^0 - k^0) + \; \vec{\gamma} \; \cdot \; \vec{k} - \; m\mathbf{1} \; + \; i\epsilon}\right]^{(2)} \; . \end{aligned} \tag{2.26}$$

Eq. (2.25) is a 4-dimensional integral equation. Even after partial wave decomposition it is still 2-dimensional, thus more complicated to solve. As a technical detail, one assumes for the solution that the particles in the final state are not on-mass shell and picks then the physical solution. (Compare solution of Lippmann-Schwinger equation.)

2.6 Structure of the Two-Nucleon Propagator

One has according to (2.10)

$$S(p) \;=\; \frac{\gamma^\nu p_\nu \;+\; m\mathbf{1}}{p^2 \;-\; m^2 \;+\; i\epsilon}\;, \tag{2.27}$$

where in general $p^0 \neq E_p = \sqrt{p^2 + m^2}$. Introduce projection operators on particles 4 × 4 matrices):

$$\Lambda_+(p) = \sum_{i=1}^{2}\; u^{(i)}(p)\bar{u}^{(i)}(p) \;=\; \frac{1}{2m}\,(\gamma^0 E_p \;-\; \vec{\gamma}\cdot\vec{p} \;+ m\mathbf{1}) \tag{2.28}$$

and on anti-particles

$$\Lambda_-(p) \;=\; -\sum_{i=1}^{2}\; v^{(i)}(p)\;\bar{v}^{(i)}(p) \;=\; \frac{1}{2m}\,(-\gamma^0 E_p \;+ \vec{\gamma}\cdot\vec{p} \;+\; m\mathbf{1}) \tag{2.29}$$

where $v^{(1)}(p) \;=\; u^{(3)}(-p)$ and $v^{(2)}(p) \;=\; u^{(4)}(-p)$. [For notation, see Bjorken-Drell.] The projection operators have the following properties:

$$\begin{aligned} (\Lambda_\pm)^2 \;&=\; \Lambda_\pm \\ \Lambda_+ \;+\; \Lambda_- \;&=\; \mathbf{1} \\ \Lambda_+ \;\cdot\; \Lambda_- \;&=\; 0 \end{aligned} \tag{2.30}$$

With

$$\Lambda_-(-p) \;=\; -\sum_{i=1}^{2}\; v^{(i)}(-p)\bar{v}^{(i)}(-p) \;=\; \frac{1}{2m}\,(-\gamma^0 E_p \;- \vec{\gamma}\cdot\vec{p} \;+\; m\mathbf{1}) \tag{2.31}$$

follows

$$\begin{aligned} \Lambda_+(p) \;-\; \Lambda_-(-p) \;&=\; \frac{E_p}{m}\,\gamma^0 \\ \Lambda_+(p) \;+\; \Lambda_+(-p) \;&=\; \frac{1}{m}\,(-\vec{\gamma}\cdot\vec{p} \;+\; m\mathbf{1}) \end{aligned} \tag{2.32}$$

Consider the numerator of (2.10):

$$\begin{aligned}\gamma^\nu p_\nu + \ m\mathbf{1} \quad &= \ m\,\frac{p^0}{E_p}\cdot\ \frac{E_p}{m}\,\gamma_0\ +\ m\,\frac{-\vec{\gamma}\cdot\vec{p}\ +\ m\mathbf{1}}{m} \\ &= \ \frac{mp^0}{E_p}\Big[\Lambda_+(p)\ -\ \Lambda_-(-p)\Big]\ +\ m\Big[\Lambda_+(p)\ +\ \Lambda_-(-p)\Big] \\ &= \ \frac{m}{E_p}\Big[(p^0+E_p)\Lambda_+(p)\ -\ (p^0-E_p)\Lambda_-(-p)\Big]\end{aligned}$$

Thus the relativistic single nucleon propagator is given by

$$S(p)\ =\ \frac{\gamma^\nu p_\nu\ +\ m\mathbf{1}}{p^{0^2}\ -\ E_p^2\ +\ i\epsilon}\ =\ \frac{m}{E_p}\left[\frac{\Lambda_+(p)}{p^0-E_p\ +\ i\epsilon}\ -\ \frac{\Lambda_-(-p)}{p^0+E_p-i\epsilon}\right] \tag{2.33}$$

The poles are at $p^0 = E_p$ (particles) and $p^0 = -E_p$ (anti-particles). The two-nucleon propagator contains particles and anti-particles, thus there are four different possibilities for the intermediate states.

$$\begin{aligned}G(k,p^0)\ &=\ \left[\frac{1}{\gamma^0(\frac{1}{2}P^0+k^0)-\vec{\gamma}\cdot\vec{k}-m\mathbf{1}+i\epsilon}\right]^{(1)}\left[\frac{1}{\gamma^0(\frac{1}{2}P^0+k^0)\ +\ \vec{\gamma}\cdot\vec{k}-\ m\mathbf{1}\ +\ i\epsilon}\right]^{(2)} \\ &=\ \frac{m^2}{E_k^2}\left[\frac{\Lambda_+(k)}{\frac{1}{2}\ P^0+k^0\ -\ E_k+i\epsilon}\ -\ \frac{\Lambda_-(-k)}{\frac{1}{2}\ P^0+k^0\ +\ E_k\ -\ i\epsilon}\right]^{(1)} \\ &\quad\times\left[\frac{\Lambda_+(-k)}{\frac{1}{2}P^0-k^0\ -\ E_k+i\epsilon}\ -\ \frac{\Lambda_-(k)}{\frac{1}{2}P^0-k^0+E_k-i\epsilon}\right]^{(2)} \\ &=\ \frac{m^2}{E_k^2}\sum_{i,j=1}^{2}\left[\frac{u^{(i)}(\vec{k})\ \bar{u}^{(i)}(\vec{k})}{\frac{1}{2}P^0+k^0-E_k+i\epsilon}\ +\ \frac{v^{(i)}(-\vec{k})\ \bar{v}^{(i)}(-\vec{k})}{\frac{1}{2}P^0+k^0+E_k-i\epsilon}\right]^{(1)} \\ &\quad\times\left[\frac{u^{(j)}(-\vec{k})\ \bar{u}^{(j)}(-\vec{k})}{\frac{1}{2}P^0-\ k^0-E_k+i\epsilon}\ +\ \frac{v^{(j)}(\vec{k})\ \bar{v}^{(j)}(\vec{k})}{\frac{1}{2}P^0-k^0+E_k-i\epsilon}\right]^{(2)}\end{aligned} \tag{2.34}$$

2.7 One-Pion Exchange

Consider the Born term:

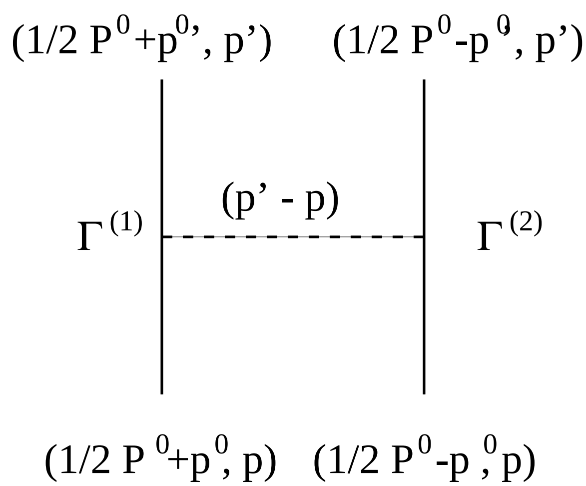

Figure 2.2.5: One meson exchange

with the vertices in pseudoscalar coupling $\Gamma^{(i)} = \sqrt{4\pi}\ g_{ps}\ i\gamma^5\ \tau_j^{(i)}$. In this case, one obtains for $\mathbf{K}(p', p \mid P^0)$:

$$
\begin{aligned}
\hat{\mathbf{K}}(p', p \mid \ P^0) \quad &= \quad \frac{1}{(2\pi)^3}\ \frac{\Gamma^{(1)}\ \Gamma^{(2)}}{(p'-p)^2 - m_{ps}^2 + i\epsilon} \\
&= \quad -\frac{4\pi}{(2\pi)^3}\ g_{ps}^2 \gamma^5 \gamma^5 \left[(p^{0'} - p^0)^2 - (\vec{p}\,' - \vec{p})^2 - m_{ps}^2 + i\epsilon \right]^{-1}\ \tau^{(1)} \cdot \tau^{(2)}\ .
\end{aligned}
\tag{2.35}
$$

Both $G(k, P^0)$ and $\mathbf{K}(p', p \mid P^0)$ are 16×16 matrices. Thus, the Bethe-Salpeter equation is in its original form a (16×16) matrix equation. Consider matrix elements. For positive energy, one obtains

$$
\begin{aligned}
\mathbf{K}^{++,++}_{s_1' s_2' s_1 s_2}\ (p', p \mid P^0) \quad &= \quad \bar{u}(p', s_1')\ \bar{u}(-p' s_1')\ \hat{\mathbf{K}}(p', p \mid P^0)\ u(p_1 s_1) u(-p, s_2) \\
&= \quad \frac{-4\pi}{(2\pi)^3}\ g_{ps}^2\ \frac{\bar{u}(p' s_1')\ \gamma^5\ \bar{u}(-p_1' s_2')\ \gamma^5\ u(-p_1 s_2)}{(p_0' - p_0)^2 - (\vec{p}\,' - \vec{p})^2 - m_{ps}^2 + i\epsilon}
\end{aligned}
\tag{2.36}
$$

which is valid for free particles. The indices s_i indicates the spin degrees of freedom. Similarly, one obtains

$$
\mathbf{K}^{+-,++}_{s_2' s_2' s_1 s_2}\ (p' p \mid P^0) \ = \ \bar{u}(p' s_1')\ \bar{v}(p' s_2') \hat{\mathbf{K}}(p' p \mid P^0)\ u(p, s_1) u(p, s_2) \tag{2.37}
$$

as well as all other matrix elements.

One can now write down the Bethe-Salpeter equation for matrix elements. If one is interested in NN-scattering, one needs only the positive energy solutions in the initial and final state. However, in the intermediate data all solutions can appear. Consider

$$\begin{aligned} M^{++,++}_{s_1's_2's_1s_2}(q'q \mid P^0) &= \mathbf{K}^{++,++}_{s_1's_2's_1s_2}\ (q_1'q \mid P^0) \\ &+ \sum_{s_1''s_2''} \frac{1}{(2\pi)^4} \int d^4k \frac{m^2}{E_k^2} \Big[\mathbf{K}^{++,++}_{s_1's_2's_1''s_2''}(q_1'k \mid P^0) G^{++}(k;P^0) M^{++,++}_{s_1''s_2'',s_1s_2}(k,q \mid P^0) \\ &+ \quad \mathbf{K}^{++,+-}\ G^{+-}\ M^{+-,++}\ +\ \mathbf{K}^{++,-+}\ G^{-+}\ M^{-+,++} \\ &\quad + \mathbf{K}^{++,--}\ G^{--}\ M^{--,++} \Big] \end{aligned} \tag{2.38}$$

with

$$G^{\alpha\beta}(k,P^0) \ = \ \frac{1}{k^0 \ - \ \omega_1^\alpha} \frac{1}{k^0 \ - \ \omega_2^\beta} \tag{2.39}$$

and

$$\begin{aligned} \omega_1^+ &= E_k - \frac{1}{2} P^0 - i\epsilon \\ \omega_1^- &= -E_k - \frac{1}{2} P^0 + i\epsilon \\ \omega_2^+ &= -E_k + \frac{1}{2} P^0 + i\epsilon \\ \omega_2^- &= E_k + \frac{1}{2} P^0 - i\epsilon \end{aligned} \tag{2.40}$$

Location of the poles:

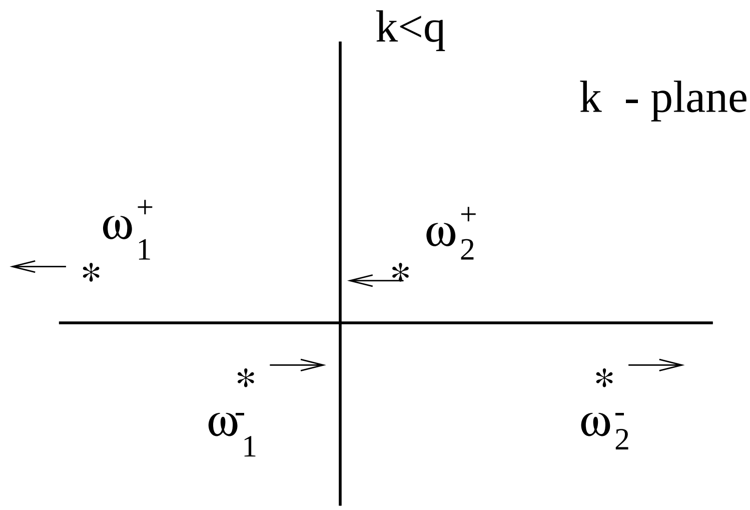

Figure 2.2.6: Location of the poles of the 2-nucleon propagator

Eq. (2.38) is a highly coupled system of integral equations for the different contributions to $M(q'q \mid P^0)$, and for its solution the knowledge of all $M^{\alpha\beta,\gamma^\delta}$ is required. A simplification can be reached when the anti-particle contributions can be neglected. It has been seen that the anti-particle contributions are large for the pseudoscalar coupling $i\gamma^5$; however, they are small for the pseudovector coupling $(i\gamma^5\ \gamma^\mu\ \Delta_\mu)$. One believes that anti-particle contributions are small, thus the preferred coupling for the pion is pseudovector coupling. A technical difficulty of pseudovector coupling is that one obtains additional contact terms when constructing a Hamiltonian, and these contact terms need to be considered.

If anti-particle contributions are neglected, the equation for the scattering amplitude (2.38) reduces to

$$M^{++,++} = \mathbf{K}^{++,++} + \mathbf{K}^{++,++}\, G^{++}\, M^{++,++} \tag{2.41}$$

2.8 Reductions of the Bethe-Salpeter Equation

The starting point is the original Bethe-Salpeter equation

$$M \;=\; \mathbf{K} \;+\; \mathbf{K}GM \tag{2.42}$$

Introduce a simpler propagator g

$$M \;=\; W \;+\; WgM \tag{2.43}$$

$$W \;=\; \mathbf{K} \;+\; \mathbf{K}(G-g)W \tag{2.44}$$

Here we only rewrote (2.42) into a system of 2 coupled equations. Proof by insertion

$$\begin{aligned} M &= W \;+\; WgM \\ &= \mathbf{K} \;+\; \mathbf{K}GW \;-\; \mathbf{K}gW \;+\; (\mathbf{K} \;+\; \mathbf{K}(G-g)W)\; gM \\ &= \mathbf{K} \;+\; KGW \;+\; \mathbf{K}G\,Wg\;M \;+\; \mathbf{K}gM \;-\; \mathbf{K}gW \;-\; \mathbf{K}g\;Wg\;M \\ &= \mathbf{K} \;+\; \mathbf{K}GM \;+\; \mathbf{K}gM \;-\; \mathbf{K}g\;M \\ &= \mathbf{K} \;+\; \mathbf{K}GM \end{aligned}$$

The two systems of equations (2.42) and (2.43), (2.50) are equivalent, and due to (2.44), the second system is as complicated as the first. The requirement for g should be that it is simple, and the difference $(G-g)$ should be small, so that (2.44) can be simplified as

$$W \;\approx\; \mathbf{K} \;+\; \mathbf{K}(G-g)K \tag{2.45}$$

which is no longer an integral equation. $\mathbf{K}$ represented the sum of infinitely many diagrams

$$\mathbf{K} \;=\; \mathbf{K}^{(2)} \;+\; \mathbf{K}^{(4)} \;+\; \cdots \tag{2.46}$$

same is

$$W \;=\; W^{(2)} \;+\; W^{(4)} \;+\; \cdots \tag{2.47}$$

The first terms are identical, i.e., $\mathbf{K}^{(2)} \;=\; W^{(2)} \;\equiv$ one-pion exchange (e.g.). Then

$$W^{(4)} \;=\; \mathbf{K}^{(4)} \;+\; \mathbf{K}^{(2)}\,G\mathbf{K}^{(2)} \;-\; \mathbf{K}^{(2)}\,g\mathbf{K}^{(2)}\;. \tag{2.48}$$

For $\mathbf{K}^{(4)}$ being the cross diagram, we obtain diagrammatically

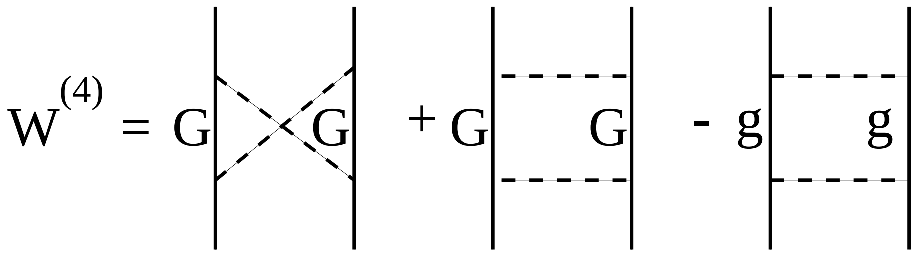

Figure 2.5.1 Diagrammatic representation of $W^{(4)}$

If $\mathbf{K}^{(4)}$ is included (very important with γ^5-coupling), then $W^{(4)}$ is small if g is a reasonable choice of propagator. Under these circumstances, one would expect a good convergence of the expansion.

Determination of g:

The requirement for g should be:

(a) g should be simple.

(i) g should contain only particle states, i.e.,

$$g(k, P^0) \;=\; g'(k, P^0)\; \Lambda_+(k)\; \Lambda_+(-k)\;, \tag{2.49}$$

then

$$\begin{aligned} M^{++,++}_{s_1',s_2',s_1s_2}\;(q'q \mid P^0) \;&=\; W^{++,++}_{s_1',s_2',s_1s_2}\;(q',q \mid P^0) \\ &+\; \sum_{s_1''s_2''} \frac{1}{(2\pi)^4} \int d^4k\; W^{++,++}_{s_1's_2',s_1''s_2''}\;(q',k \mid P^0)\; G'(k,P^0) \\ &\quad M^{++,++}_{s_4''s_2''s_1s_2}(k,q \mid P^0) \end{aligned} \tag{2.50}$$

Eq. (2.50) is still four-dimensional. One would like to have a three-dimensional equation, so a possible definition of g is

(ii)

$$g'(k, P^0) \;=\; \delta(k^0 \;-\; F(\vec{k}, P^0))\; \bar{g}(\vec{k}, P^0) \tag{2.51}$$

With this the scattering amplitude reads:

$$
\begin{aligned}
M_{s_1''s_2''s_1s_2}(\vec{q}\,',\vec{q}\mid P^0) &= W^{++,++}_{s_1's_2's_1s_2}(\vec{q}\,',\vec{q}\mid P^0) \\
&= \sum_{s_1''s_2''}\frac{1}{(2\pi)^4}\int d^3k\; W^{++,++}_{s_1's_2s_1''s_2''}\;(\vec{q},\vec{k}\mid P^0)\;\bar{g}(\vec{k},P^0) \\
&\quad M^{++,++}_{s_1''s_2''s_1s_2}(\vec{k}\vec{q}\mid P^0) \qquad (2.52)
\end{aligned}
$$

Eq. (2.52) has the derived three-dimensional form. One has to determine $F(\vec{k},P^0)$ and $G(\vec{k},P^0)$, which should be chosen such that the above requirement on G are fulfilled.

(b) g should closely represent G, i.e. $(G-g)$ should be small:

(i) $\bar{g}$ has to be chosen such that M fulfills the relativistic unitarity condition.

(ii) $\bar{G}$ is not uniquely determined by (i). A further choice has to be made for $F(K,P^0)$ in $G'(K,P^0)$. In principle, there are infinitely many possible choices.

Several suggestions are given in the literature:

Blankenbecler-Sugar (BbS, 1966):

$$
\begin{aligned}
g'(k,P^0) &= \delta(k^0)\;2\pi\;\frac{m^2}{E_k}\;\frac{1}{\frac{1}{4}\;P^{0\;2}\;-\;E_k^2+i\epsilon} \qquad (2.53)\\
&= \delta(k^0)\;2\pi\;\frac{m^2}{E_k}\;\frac{1}{q^2\;-\;k^2\;+\;i\epsilon} \qquad (2.54)
\end{aligned}
$$

For the last relation $P^0 = 2E_q$ was used. The BbS choice looks as the non-relativistic Lippmann-Schwinger equation.

Erkelenz-Holinde (EH, 1973):

$$
g'(k,P^0) \;=\; \delta(k^0\;-\;E_k\;+\;\frac{1}{2}\;P^0)\;2\pi\;\frac{m^2}{E_k}\;\frac{1}{q^2\;-\;k^2\;+\;i\epsilon} \qquad (2.55)
$$

The scattering amplitude is given then as

$$
\begin{aligned}
M^{BbS,EH}_{s_1's_2's_1s_2}\;(\vec{q}\,',\vec{q}\mid P^0) &= W^{BbS,EH}_{s_1's_2's_1s_2}\;(\vec{q}',\vec{q}\mid P^0) \\
&+ \sum_{s_1''s_2''}\frac{m}{(2\pi)^3}\int d^3k\;\frac{m}{E_k}\;W^{BbS,EH}_{s_1's_2's_1''s_2''}\;(\vec{q}\,',\vec{k}'\mid P^0) \\
&\times \frac{1}{q^2\;-\;k^2+i\epsilon}\;M^{BbS,EH}_{s_1''s_2''s_1s_2}\;(\vec{k},\vec{q}\mid P^0) \qquad (2.56)
\end{aligned}
$$

Due to the factor m/E_k, the equation (2.56) is still covariant. For e.g., the one-pion exchange from Sect. 2.4 one has the following consequences:

$$\mathbf{K}_{ps}^{BS}\ (p'p \mid P^0)\ =\ -\frac{4\pi}{(2\pi)^3}\ g_{ps}^2 \frac{\bar{u}(p's_1')\ \gamma^5\ u(p_1s_1)\ \bar{u}(-p's_2')\gamma^5\ \bar{u}(-ps_2)}{(p^{0'}\ -\ p^0)^2\ -\ (\vec{p}\ '-\vec{p})^2 - m_{ps}^2 + i\epsilon} \tag{2.57}$$

BbS-Choice: $P^{0\prime} = P^0 \implies P_1^0 = P_2^0$ and $P_1^{\prime 0} = P_2^{\prime 0}$, i.e., both particles in the intermediate state are equally for off-mass shell. Or in other words, there is only three-momentum transfer by the exchanged particles, no energy transfer, and $(P^{0\prime} - P^0)^2 = 0$.

EH-Choice: $P^{0\prime} = E_p' - \frac{1}{2}$; $P^0 = E_p - \frac{1}{2}P^0 \implies (P^{0\prime} - P^0)^2 = (E_p' - E_p)^2$. Thus a retardation is included in the meson propagator. Particle 1 is on-mass shell, particle 2 in general not.

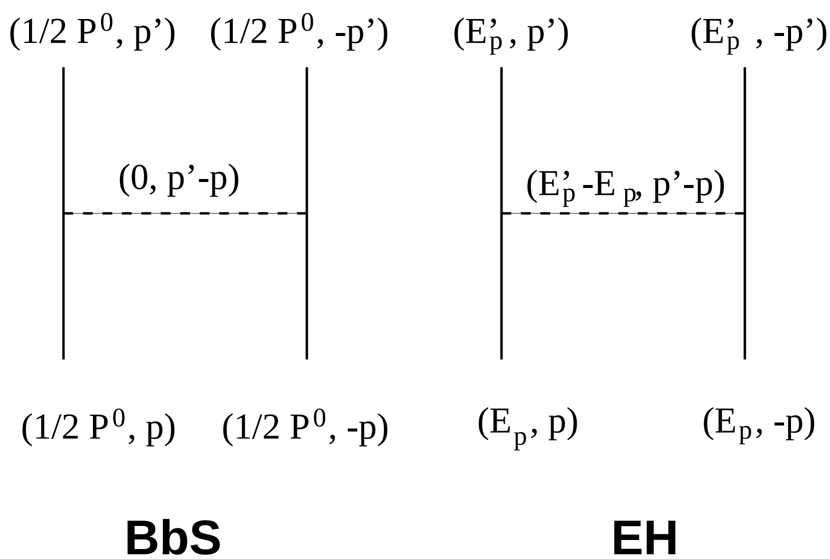

Figure 2.5.2: Illustrations of different choices for $G'(k, P^0)$

For simplicity, spin degrees of freedom shell be omitted in the following consideration. After making the favorite choice for $G'(K, P^0)$, one has the following three-dimensional integral equation for M:

$$M(\vec{q}\,', \vec{q})\ =\ W(\vec{q}\,', \vec{q})\ +\ \frac{m}{(2\pi)^3}\ \int d^3k\ W(\vec{q}\,', \vec{k})\ \sqrt{\frac{m}{E_k}}\ \frac{1}{q^2 - k^2 + i\epsilon}\ \sqrt{\frac{m}{E_k}}\ M(\vec{k}, \vec{q}) \tag{2.58}$$

multiplication from the left with $\sqrt{\frac{m}{E_{q'}}}$ and from the right with $\sqrt{\frac{m}{E_q}}$ gives

$$T(\vec{q}\,',\vec{q}) \;=\; V(\vec{q}\,',\vec{q}) \;+\; \frac{m}{(2\pi)^3} \int d^3k\; V(\vec{q}',\vec{k})\; \frac{1}{q^2 - k^2 + i\epsilon}\; T(\vec{k},\vec{q}) \;, \tag{2.59}$$

which has the form of a nonrelativistic Lippmann-Schwinger equation with the potential

$$V(\vec{q}\,',\vec{q}) \;=\; \sqrt{\frac{m}{E_{q\,'}}}\; W(\vec{q}',\vec{q})\; \sqrt{\frac{m}{E_q}} \;. \tag{2.60}$$

The factors $\sqrt{\frac{m}{E_q}}$ are often called "minimal relativity factors," and they have their justification through the derivation (2.58). From the t-matrix, one can in the usual way proceed to the K-matrix. If the potential is real, this is the logical choice. One has the Heitler equation

$$T(E) \;=\; \mathbf{K}(E) \;-\; i\pi\; K(E)\; \delta(E \;-\; H_0)\; T(E) \tag{2.61}$$

and obtains

$$K(E) \;=\; V \;+\; V\; \mathbf{P}\; \frac{1}{E - H_0}\; K(E) \;. \tag{2.62}$$

In partial waves: $T_\ell \;\approx\; e^{i\delta_\ell} \sin\delta_\ell$ and $K_\ell \;\approx \tan\delta_\ell$.

2.9 Time-Ordered Perturbation Theory

The technique that has historically been most useful in calculating the S-matrix is perturbation theory, an expansion in powers of the interaction term V in a Hamiltonian $H \;=\; H_0 \;+\; V$, where H_0 is the free Hamiltonian. The S-matrix can be written as

$$S_{\alpha\beta} \;=\; \delta(\beta \;-\; \alpha) \;-\; 2i\pi\; \delta(E_\beta \;-\; E_\alpha) T^+_\mu \tag{2.63}$$

with

$$T^+_{\beta\alpha} \;=\; \langle\; \phi_\beta \;\mid\; V\; \psi^+_\alpha \rangle \;. \tag{2.64}$$

Here α, β characterize the initial, final states, $\mid \phi_\beta\rangle$ is a free state (solution to H_0) and $\mid \psi^+_\alpha\rangle$ a scattering state satisfying a Lippmann-Schwinger equation

$$\mid \psi^+_\alpha\rangle \;=\; \phi_\alpha \;+\; \int d\gamma\; \frac{T_{\gamma\alpha}\phi_\gamma}{E_\alpha \;-\; E_\gamma \;+\; i\epsilon} \;. \tag{2.65}$$

Multiplying with V and taking the scalar product with $\langle \phi_\beta \mid$ gives the standard form of the operator $L - S$ equation

$$T^+_{\beta\alpha} \;=\; V_{\beta\alpha} \;+\; \int d\gamma \, \frac{V_{\beta\gamma}\, T^+_{\gamma\alpha}}{E_\alpha \;-\; E_\gamma \;+\; i\epsilon} \tag{2.66}$$

where $V_{\beta\alpha} \;\equiv\; \langle \phi_\beta \mid V \mid \phi_\alpha \rangle$. The perturbation series for $T^+_{\beta\alpha}$ is obtained by iterating (2.66) as

$$\begin{aligned} T^+_{\beta\alpha} \;&=\; V_{\beta\alpha} \;+\; \int d\gamma \, \frac{V_{\beta\alpha}\, V_{\gamma\alpha}}{E_\alpha \;-\; E_\gamma \;+\; i\epsilon} \\ &+\; \int d\gamma \, d\gamma' \, \frac{V_{\beta\gamma} V_{\gamma\gamma'}\, V_{\gamma'\alpha}}{(E_\alpha \;-\; E_\gamma \;+\; i\epsilon)\,(E_\alpha \;-\; E'_\gamma \;+\; i\epsilon)} \;+\; \cdots \end{aligned} \tag{2.67}$$

This method of calculating the S-matrix is today called **old-fashioned- perturbation theory**. Its obvious drawback is that the energy denominators obscure the underlying Lorentz invariance of the S-matrix. A rewritten version of (2.67) is known as **time-dependent perturbation theory**. This has the virtue of making the Lorentz structure more obvious, while somewhat obscuring the contribution of the individual intermediate states. The time-ordered perturbation expansion can be derived from the S-matrix in the form

$$S \;\equiv\; U(\infty, -\infty) \;, \tag{2.68}$$

where

$$U(\tau, \tau_0) \;:=\; e^{iH_0\tau} \; e^{-iH(\tau-\tau_0)} \; e^{-iH_0\tau_0} \tag{2.69}$$

Differentiating (2.69) with respect to τ gives

$$i \, \frac{d}{d\tau} \, U(\tau, \tau_0) \;=\; V(\tau) \; U(\tau_1 \tau_0) \tag{2.70}$$

where

$$V(t) \;=\; e^{iH_0 t} \; V \; e^{oH_0 t} \;, \tag{2.71}$$

which corresponds to the definition of time dependence for an operator in the interaction picture. Eq (2.70) together with the initial condition $U(\tau_0, \tau_0) \;=\; \mathbf{1}$ is satisfied by the solution

$$U(\tau, \tau_0) \;=\; \mathbf{1} \;-\; i \int_{\tau_0}^{\tau} dt \; V(t) \; U(t, \tau_0) \;. \tag{2.72}$$

By iterating this integral equation, one obtains an expansion for $U(\tau, \tau_0)$ in powers of V:

$$\begin{aligned} \mu(\tau, \tau_0) \;&=\; \mathbf{1} \;-\; i \int_{\tau_0}^{\tau} dt_1 \; V(t_1) \;+\; (-i)^2 \int_{\tau 0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 \; V(t_1) \; V(t_2) \\ &+\; (-i)^3 \int_{\tau_0}^{\tau} dt_1 \int_{\tau 0}^{t_1} dt_2 \int_{\tau_0}^{t_2} dt_3 \; V(t_1) \; V(t_2) \; V(t_3) \;+\; \cdots \end{aligned} \tag{2.73}$$

Setting $\tau \equiv \infty$ and $\tau_0 \equiv -\infty$ gives the perturbation expansion for the S-operator

$$S = \mathbf{1} - i \int_{-\infty}^{\infty} dt_1 \, V(t_1) + (-i)^2 \int_{\infty}^{\infty} dt_1 \int_{\infty}^{t_1} dt_2 \, V(T_1) \, V(t_2) + \cdots \tag{2.74}$$

This can also be derived directly from (2.67) by using the Fourier representation of the energy factors in (2.67)

$$(E_\alpha - E_\gamma + i\epsilon)^{-1} = -i \int_0^{\infty} d\tau \, e^{i(E_\alpha - E_\gamma)\tau} \tag{2.75}$$

with the understanding that such integrals are to be evaluated by inserting a convergence factor $e^{\epsilon\tau}$ in the integrand with $\epsilon \to 0^+$.

One can rewrite (2.74) in a way that proves very useful in carrying out manifestly Lorentz-covariant calculations. For this, define the **time-ordered product** of any time-dependent operators as the product with factors arranged so that the one with the latest time argument is placed leftmost, the next latest next to the leftmost, etc.

$$T\{V(t)\} = V(t) \tag{2.76}$$

$$T\{V(t_1) \, V(t_2)\} = \theta(t_1 - t_2) \, V(t_1) \, V(t_2) + \theta(t_2 - t_1) \, V(t_2) \, V(t_1) \, , \tag{2.77}$$

where $\theta(\tau)$ is the step function equal $+1$ for $\tau > 0$, zero for $\tau < 0$. Then the time-ordered product of n V's is the sum over all $n!$ permutations of the V's, each of which gives the same integral over all $t_1 \cdots t_n$. Thus Eq. (2.74) may be written

$$S = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n)}{n!} \int_{-\infty}^{\infty} dt_1 \, dt_2 \cdots dt_n \; T\{V(t_1) \cdots V(t_n)\} \, . \tag{2.78}$$

This is sometimes known as **Dyson series**. If the $V(t)$ at different times all commute, the series can be summed up as

$$S = \exp \left[-i \int_{-\infty}^{\infty} dt \, V(t) \right] \, . \tag{2.79}$$

Of course, this is usually **not** the case – (2.78) does in general not converge.

One can now find a class of theories for which the S-matrix is manifestly Lorentz invariant. Since the elements of the S-matrix are the matrix elements of S taken between free states, ϕ_α, ϕ_β, the S-operator should commute with the operator $U_0(\Lambda, a)$, which produces Lorentz transformations on free states. Equivalently, S must commute with the generators of $\mu_0(\Lambda, a)$, namely $H_0, \vec{P}_0, \vec{J}_0$, and $\vec{K}_0$.

To satisfy this requirement, assume that $V(t)$ is given as

$$V(t) \;=\; \int d^3x \; \mathbf{H}(t, \vec{x}) \tag{2.80}$$

with $\mathbf{H}(x)$ being a scalar in the sense that

$$U_0(\Lambda, a) \; \mathbf{H}(x) \; U_0^{-1}(\Lambda, a) \;=\; \mathbf{H}(\Lambda x + a) \; . \tag{2.81}$$

Everything is now manifestly Lorentz invariant, except for the time ordering of the operator product. The time ordering of two space-time points x_1, x_2 is Lorentz invariant unless $x_1 - x_2$ is space-like, i.e., $(x_1 - x_2)^2 > 0$, thus the time ordering in (2.81) introduces no special Lorentz frame if the $\mathbf{H}(x)$ commute at space-like or light-like separations

$$[\mathbf{H}(x), \; \mathbf{H}(x')] \;=\; 0 \qquad \text{for} \;\; (x - x')^2 \;>\; 0 \; . \tag{2.82}$$