H7.1 **Pauli Matrices I** (2P)

Establish the identity

\[
(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma}(\vec{A} \times \vec{B})
\]

where \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) are the Pauli matrices, and \(\vec{A}\) and \(\vec{B}\) are vector operators which commute with \(\vec{\sigma}\) but not necessarily commute with each other.

H7.2 **Pauli Matrices II** (2P)

Consider an arbitrary matrix

\[
M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.
\]

Show that the matrix \(M\) can always be written as a linear combination of the form matrices \(I, \sigma_x, \sigma_y, \sigma_z\)

\[
M = a_0 I + \vec{\alpha} \cdot \vec{\sigma}
\]

where \(a_0, a_x, a_y,\) and \(a_z\) are complex numbers.

H7.3 **Coupling of Spin Operators** (2P)

Let \(\vec{s}_1\) and \(\vec{s}_2\) be the spin operators of two spin \(\frac{1}{2}\) particles. Find the simultaneous eigenfunctions of the operators \(\vec{s}^2\) and \(s_z\), where \(\vec{s} = \vec{s}_1 + \vec{s}_2\). Show that these are also eigenfunctions of the operator \(\vec{s}_1 \cdot \vec{s}_2\).

H7.4 **Clebsch-Gordan Coefficients** (4P)

a) Prove the following recursion relation for the Clebsch-Gordan coefficients

\[
\sqrt{j(j + 1) - m(m \pm 1)} \langle j_1, m_1; j_2, m_2 | j, m + 1 \rangle = \sqrt{j_1(j_1 + 1) - m_1(m_1 \mp 1)} \langle j_1, m_1 \mp 1; j_2, m_2 | j, m \rangle + \sqrt{j_2(j_2 + 1) - m_2(m_2 \mp 1)} \langle j_1, m_1; j_2, m_2 \mp 1 | j, m \rangle
\]

(Hint: use \(\langle j_1, m_1; j_2, m_2 | J_\pm | j, m \rangle\).)

b) Let \(j_1 = \frac{1}{2}\) and \(j_2 = l\). Using a) compute

\[
\langle \frac{1}{2}, \pm \frac{1}{2}; l, m \mp \frac{1}{2} | l + \frac{1}{2}, m \rangle = \sqrt{\frac{l \pm m + \frac{3}{2}}{2l + 1}}
\]

\[
\langle \frac{1}{2}, \pm \frac{1}{2}; l, m \mp \frac{1}{2} | l - \frac{1}{2}, m \rangle = \pm \sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}}.
\]