Phys. 735: Homework II

due September 22, 2006

1. An operator $U(t+\varepsilon,t)$ describes the change in a wave function from t to $t+\varepsilon$. For ε real and small enough so that terms $\sim \varepsilon^2$ may be neglected

$$U(t+\varepsilon,t) = \mathbf{1} - \frac{i}{\hbar}\varepsilon H(t).$$

(a) *(3 pts)*
Show that U is unitary, if and only if H is hermitian.

(b) *(2 pts)*
Show that an alternate form

$$U(t+\varepsilon,t) = \frac{\mathbf{1} - \frac{i}{2\hbar}\varepsilon H(t)}{\mathbf{1} + \frac{i}{2\hbar}\varepsilon H(t)}$$

agrees with the U given above (provided terms of order ε^2 can be neglected) and is unitary if H is hermitian.

2. Consider a particle of mass m in the complex potential field

$$\Phi = V(\mathbf{r}) + \frac{i\hbar}{2}\omega(\mathbf{r}),$$

where $V(\mathbf{r})$ and $\omega(\mathbf{r})$ are real functions.

(a) *(4 pts)*
What form does the continuity equation assume for this potential?

(b) *(2 pts)*
Offer an interpretation for the field $\omega(\mathbf{r})$.

(c) *(2 pts)*
Is the new continuity equation found in (a) time-reversible? (Prove your answer).

(d) *(2 pts)*
Is the related Hamiltonian Hermitian? (Prove your answer).

3. *(10 pts)*
For the simple harmonic oscillator the wave function of an energy eigenstate is given by

$$u_n(x) \exp\left(-\frac{i}{\hbar}E_n t\right) = c_n \exp\left(-\frac{1}{2}\alpha^2 x^2\right) H_n(\alpha x),$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}} \; ; \quad c_n^2 = \frac{1}{2^n\, n!}\frac{\alpha}{\sqrt{\pi}} \; ; \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Determine the propagator $\mathcal{K}(x'', t; x', t_0)$, where you may consider $t_0 = 0$ for simplicity. Discuss the time behavior of the wave function.
Potentially useful formulae:

$$\exp\left(-\lambda^2 + 2\lambda\eta\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(\eta)$$

$$\left(\frac{1}{\sqrt{1-\gamma^2}}\right) \exp\left(\frac{-(\alpha^2 + \beta^2 - 2\alpha\beta\gamma)}{(1-\gamma^2)}\right) = \exp[-(\alpha^2 + \beta^2)] \sum_{n=0}^{\infty} \left(\frac{\gamma^n}{2^n n!}\right) H_n(\alpha) H_n(\beta)$$