Phys. 735: Homework III

due September 29, 2006

1. *(5 pts)*
Define a set of mappings £: $M^4 \to M^4$: $\forall L \in \pounds$, there is $g = LgL^T$
where M^4 is Minkovski space with following metric g

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Prove that $\pounds$ forms a group.

2. *(5pts)*
Write the Lorentz transformation in terms of rapidity R:

$$\begin{bmatrix} x_0' \\ x_1' \end{bmatrix} = \begin{bmatrix} \cosh R & -\sinh R \\ -\sinh R & \cosh R \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

where $\gamma = \cosh R$ and $\beta = \tanh R$.
Prove that

$$exp(-R\sigma_1) = \begin{bmatrix} \cosh R & -\sinh R \\ -\sinh R & \cosh R \end{bmatrix}$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hint: Refer the related matrix algebra for calculating e^A, with A being a matrix

3. *(10 pts)*
3.1 *(4 pts)*
Prove that in M^4, the boost along the x direction L_x can be written in terms of the exponentiation of generator L_1:

$$L_x = exp(-R_x L_1)$$

where $R_x = \tanh(v_x/c)$ and L_1 is the extension of σ_1 written as

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2 *(3 pts)*
By the example of 3.1, find the generators L_2 and L_3 which are also natural extension of σ_1, which can generate the boost along y and z direction respectively via exponentiation.

3.3 *(3 pts)*
Verify that the commutator $[L_1, L_2]$ can generate a rotation along a certain direction and in a certain plane via exponentiation. Find the direction of rotation and the plane.