Phys. 735: Homework IV

due October 11, 2006

1. The minimum coupling of the electromagnetic field is written in a four-dimensional way as

$$\hat{p}^\mu \rightarrow \hat{p}^\mu - \frac{e}{c} A^\mu$$

With this, the Klein-Gordon equation with an electromagnetic field is given as

$$(\hat{p}^\mu - \frac{e}{c} A^\mu)(\hat{p}_\mu - \frac{e}{c} A_\mu)\psi = mc^2\psi$$

(a) *[5 pts]*
Derive an expression for the four-current density in the electromagnetic field A_ν and give expressions for the charge density and the charge-current density.

Then, consider a π^- meson (with mass $m_\pi c^2$=139.577 MeV and spin 0) being bound by the Coulomb potential

$$V(r) = -\frac{Ze^2}{r}$$

in a stationary state of **total** energy E<m. A stationary state of the Klein-Gordon equation has the form

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) \exp(-iEt/\hbar)$$

where $|E|$ is the energy per particle.

(b) *[2 pts]*
What is the time-independent Klein-Gordon equation for this potential?

(c) *[4 pts]*
Assume the radial and angular parts of the wave function $\psi(\mathbf{r})$ separate. Verify that this yields

$$\frac{d^2 u_l(kr)}{dr^2} + \left[-\frac{2EZ\alpha}{r} - (m^2 - E^2) - \frac{l(l+1) - (Z\alpha)^2}{r^2} \right] u_l(kr) = 0$$

where

$$\alpha = e^2 \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

(d) *[3 pts]*
Show that this equation can be written in the dimensionless form

$$\left[\frac{d^2}{d\rho^2} - \frac{\mu^2 - 1/4}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4} \right] u_l(\rho) = 0$$

with

$$
\begin{aligned}
\rho &= \gamma r \\
\gamma^2 &= 4(m^2 - E^2) \\
\mu^2 &= \left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2 \\
\lambda &= \frac{2EZ\alpha}{\gamma}
\end{aligned} \tag{1}
$$

(e) *[3 pts]*
Assume this equation has a solution in the usual form of a power series times the $\rho \to \infty$ and $\rho \to 0$ solutions:

$$u_l(\rho) = \rho^k(1 + c_1\rho + c_2\rho^2 + c_3\rho^3 + \cdots)e^{-\rho/2} \tag{2}$$

Show that

$$k = k_\pm = \frac{1}{2} \pm \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}$$

(f) *[4 pts]*
Show that for both k_+ and k_- the wave function is divergent at the origin yet normalizable.

(g) *[4 pts]*
Show that only for k_+ is the expectation value of the kinetic energy finite:

$$\int dr\ r^2 \left[\frac{d(u_l/r)}{dr}\right]^2 < \infty$$

(h) *[4 pts]*
Show that the k_+ solution has a nonrelativistic limit which agrees with the solution found for the Schrödinger equation.

(i) *[4 pts]*
Determine the recurrence relation among the c_i's for this to be a solution of the Klein-Gordon equation.

(j) *[3 pts]*
Show that unless the power series of Eq. 1 terminates, the wave function will have an incorrect asymptotic form.

(k) *[4 pts]*
Show that the termination condition determines the eigen energy for the k_+ solution to be

$$E = m\left(1 + (Z\alpha)^2\left[n - l - \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}\right]^{-2}\right)^{1/2}$$

where n is the principal quantum number.

(l) *[4 pts]*
Expand E in powers of α^2 and show that the α^2 term yields the Bohr formula, and that higher order terms can be identified with relativistic corrections.

(m) *[3 pts]*
Is the l-degeneracy present in the nonrelativistic theory now removed? (And if so, to what order in α?)