1. Pionic Atom with a Point-Like Nucleus

The minimum coupling of the electromagnetic field is written in a four-dimensional way
as
\[ \hat{p}^\mu \rightarrow \hat{p}^\mu - \frac{e}{c} A^\mu \]  
(1)

With this, the Klein-Gordon equation with an electromagnetic field is given as
\[ (\hat{p}^\mu - \frac{e}{c} A^\mu)(\hat{p}_\mu - \frac{e}{c} A_\mu)\psi = mc^2 \psi \]  
(2)

(a) [5 pts]
Derive an expression for the four-current density in the electromagnetic field \( A_\nu \) and
give expressions for the charge density and the charge-current density.

Then, consider a \( \pi^- \) meson (with mass \( m_\pi c^2 = 139.577 \) MeV and spin 0) being bound
by the Coulomb potential
\[ V(r) = -\frac{Ze^2}{r} \]  
(3)
in a stationary state of total energy \( E < m \). A stationary state of the Klein-Gordon
equation has the form
\[ \Psi(r, t) = \psi(r) \exp(-iEt/\hbar) \]  
(4)
and \( |E| \) is the energy per particle.

(b) [2 pts]
What is the time-independent Klein-Gordon equation for this potential?

(c) [4 pts]
Assume the radial and angular parts of the wave function \( \psi(r) \) separate. Verify that
this yields
\[ \frac{d^2 u_l(kr)}{dr^2} + \left[ -\frac{2EZ\alpha}{r} - (m^2 - E^2) - \frac{l(l+1) - (Z\alpha)^2}{r^2} \right] u_l(kr) = 0 \]  
(5)

with
\[ \alpha = e^2 \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137} \]  
(6)
(d) [3 pts]

Show that this equation can be written in the dimensionless form

$$\left[ \frac{d^2}{d\rho^2} - \frac{\mu^2 - 1/4}{\rho^2} + \frac{\lambda - 1}{4} \right] u(\rho) = 0$$  \hspace{1cm} (7)

with

$$\rho = \gamma r$$
$$\gamma^2 = 4(m^2 - E^2)$$
$$\mu^2 = \left( l + \frac{1}{2} \right)^2 - (Z\alpha)^2$$
$$\lambda = \frac{2EZ\alpha}{\gamma}$$  \hspace{1cm} (8)

(e) [3 pts]

Assume this equation has a solution in the usual form of a power series times the \( \rho \to \infty \) and \( \rho \to 0 \) solutions:

$$u(\rho) = \rho^k (1 + c_1 \rho + c_2 \rho^2 + c_3 \rho^3 + \cdots) e^{-\rho/2}$$  \hspace{1cm} (9)

Show that

$$k = k_{\pm} = \frac{1}{2} \pm \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}$$  \hspace{1cm} (10)

(f) [3 pts]

Show that for both \( k_+ \) and \( k_- \) the wave function is divergent at the origin yet normalizable.

(g) [4 pts]

Show that only for \( k_+ \) is the expectation value of the kinetic energy finite:

$$\int dr \ r^2 \left[ \frac{d(u_1/r)}{dr} \right]^2 < \infty$$  \hspace{1cm} (11)

(h) [4 pts]

Show that the \( k_+ \) solution has a nonrelativistic limit which agrees with the solution found for the Schrödinger equation.

(i) [4 pts]

Determine the recurrence relation among the \( c_i \)'s for this to be a solution of the Klein-Gordon equation.
(j) [3 pts]
Show that unless the power series of (d) terminates, the wave function will have an incorrect asymptotic form.

(k) [4 pts]
Show that the termination condition determines the eigen-energy for the $k_+$ solution to be

$$E = m \left( 1 + (Z\alpha)^2 \left[ n - l - \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2} \right]^{-2} \right)^{1/2}$$

(12)

(l) [4 pts]
Expand $E$ in powers of $\alpha^2$ and show that the $\alpha^2$ term yields the Bohr formula, and that higher order terms can be identified with relativistic corrections.

(m) [3 pts]
Is the l-degeneracy present in the nonrelativistic theory now removed? (And if so, to what order in $\alpha$?)