Universality and Scaling Limit of Weakly-Bound Tetramers

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Abstract. The occurrence of a new limit cycle in few-body physics, expressing a universal scaling function relating the binding energies of two successive tetramer states, is revealed by considering a renormalized zero-range two-body interaction in bound state of four identical bosons. The tetramer energy spectrum is obtained by adding a boson to an Efimov bound state with energy $B_3$ in the unitary limit (for zero two-body binding energy or infinite two-body scattering length). Each excited $N^{th}$ tetramer energy $B_4^{(N)}$ is shown to slide along a scaling function as a short-range four-body scale is changed, emerging from the 3+1 threshold for a universal ratio $B_4^{(N)}/B_3 = 4.6$, which does not depend on $N$. The new scale can also be revealed by a resonance in the atom-trimer recombination process.

Keywords: Efimov Physics, Universality, Limit Cycle, Four-Boson Bound State, Four-Body Scale

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INTRODUCTION

In order to probe universal properties of few-boson systems at low energy, which are expected not be affected by some particular form of the interactions between particles, an appropriate simple way is by considering a renormalized zero-ranged potential. From a study of the universal scaling limit of weakly-bound three-boson system, it was found that, near the unitary limit (zero two-body energy or infinite two-body scattering length) where the Thomas-Efimov effect [1, 2, 3] is evidenced, no particular form of a two-body interaction is needed in order to obtain the correlation between the observables [4]. This universal scaling behavior, which occurs in the three-boson case, was also evidenced in Refs. [5, 6, 7], and experimentally verified with tunable two-body interactions [8, 9]. For a more detailed discussion on universal aspects of few-body systems, see Refs. [10, 11, 12, 13, 14]. In particular, Ref. [14] presents a recent review on the subject, discussing scaling behavior and universality of few-body systems.

By considering a renormalized zero-range approach for three-boson system, it was shown in Ref. [4] that the sequence of Efimov trimer states, which are numerically obtained near the unitary limit (zero two-body binding), can be represented by a single scaling plot given in figure 1 of [4]. This plot shows a scaling relation for the ratio between two three-body binding energies, as a function of the two–body energy. All the energies were given in units of a reference three-body energy, the three-body ground-state. This scaling function leads to the well-known infinite number of Efimov three-
body energies, in the limit of infinite two-body scattering length.

In view of the quite relevant role of the three-body scaling plot in revealing a limit cycle for trimer states, considering all the energies in unit of a reference three-body scale, we have consider in Ref. [15] a similar picture to study tetramer states attached to a given three-body state $B_3$, for a fixed two-body energy close to the unitary limit (near zero two-body binding).

**THE FOUR-BOSON SCALING PLOT**

The aim of our study of tetramer bound states, reported in Ref. [15], was to evidence the existence of a universal correlation between the binding energies of successive four-boson bound states for large two-body scattering lengths ($\alpha$). This correlation was shown to be related to an additional scale not constrained by three-body Efimov physics. Following an analogy with the three-body case, where a scaling function is revealing the Efimov limit cycle, we consider the results for the tetramer binding energies in a similar scaling plot. In order to obtain the tetramer energies in our renormalized zero-range approach, we have considered two scales, $\mu_3$ and $\mu_4$, for the three- and the four-boson systems, respectively, which in principle are independent and control the relation between the given trimer energy and the tetramer spectrum. Our results are summarized in Fig. 1, in case we have up to three tetramers considering the deepest bound trimer energy.

![Graph](image_url)

**FIGURE 1.** Tetramer energies (for the first three states), in units of the trimer ground state, are shown as functions of $\left(\frac{\mu_4}{\mu_3}\right)$, where $\mu_4$ and $\mu_3$ are the renormalization parameters, which define the four- and three-body momentum scales, respectively.
In this case, the tetramer results where obtained by considering a given three-body binding energy in the unitary limit (when the two-body scattering length is infinite), as well as for a few cases near this limit (with positive and negative two-body scattering lengths). With the results we have generated the universal scaling function shown in figure 1 of Ref. [15], which we reproduce in Fig. 2 of the present communication together with some clarifying explanations. Indeed, from such scaling plot, we can observe that the tetramer spectrum, for a general four-boson case, is ruled by a four-body scale detached from the three-body one.

\[ X \equiv \sqrt{B_3 / B_4^{(N)}}, \quad Y \equiv X \sqrt{B_4^{(N+1)}/B_3 - 1}. \]
By considering the same four-body scale, we can increase the number of tetramers by decreasing the ratio \(B_3/B_4^{(0)}\), such that when \(\sqrt{B_3/B_4^{(0)}} = 0.466\) the first excited state emerges at the threshold \(B_4^{(1)} = B_3\). This fixed point is given by \((X_1,Y_1) = (0.466,0)\), where \(X_1 \equiv \sqrt{B_4^{(1)}/B_4^{(0)}}\). So, when we have two tetramers (for a given \(B_3\)), they have energies given by

\[
B_4^{(0)} \simeq 4.60B_3, \quad B_4^{(1)} \simeq B_3. \tag{2}
\]

Now, we should observe that, when \(B_3 = B_4^{(2)}\) we enter the 2nd cycle having \(\sqrt{B_3} = 0.466\sqrt{B_4^{(1)}}\). From Eq. (1), we can define a straight line given by

\[
Y \equiv X\sqrt{1/X^2_1 - 1} = X\sqrt{4.60 - 1} = 1.90X. \tag{3}
\]

This relation with (1) defines the point \((X_2,Y_2)\), where \(X_2 \equiv \sqrt{B_4^{(2)}/B_4^{(0)}}\), because at this point we have \(B_3 = B_4^{(2)}\). By inspecting the plot we can find that \((X_2,Y_2) \approx (0.125,0.238)\), implying that \(B_4^{(0)} \approx 63.5B_4^{(2)}\) and \(B_4^{(0)} \approx 13.8B_4^{(4)}\).

So, when we have three tetramers, their energies are given by

\[
B_4^{(0)} \simeq 13.8B_4^{(1)}, \quad B_4^{(1)} \simeq 4.60B_4^{(2)}, \quad B_4^{(2)} \simeq B_3. \tag{4}
\]

For the next point, where \(B_3 = B_4^{(3)}\), we have \(X_2^2 = B_4^{(2)}/B_4^{(0)}\), and

\[
Y \equiv X\sqrt{B_4^{(1)}/B_4^{(3)}} - 1 = X\sqrt{1/X_2^2 - 1} = 7.91X. \tag{5}
\]

The above, together with (1) will define \((X_3,Y_3) \approx (0.014,0.111)\), implying that \(B_4^{(0)} \approx 5102B_4^{(3)}\) and \(B_4^{(0)} \approx 79.89B_4^{(1)}\). As the cyclic condition will give us \(B_4^{(2)} \simeq 4.60B_4^{(3)}\), we can obtain the spectrum for the case that \(B_4^{(3)} \simeq B_3\):

\[
B_4^{(0)} \approx 79.89B_4^{(1)}, \quad B_4^{(1)} \approx 13.88B_4^{(2)}, \quad B_4^{(2)} \approx 4.60B_4^{(3)}, \quad B_4^{(3)} \simeq B_3. \tag{6}
\]

However, there is a limit in this procedure when we consider tetramers derived for a given excited Efimov state \(B_3\). The limitation can be verified in case we are considering the unitary limit, where the ratio between two successive Efimov states is known as \(22.7^2 \sim 515\). This implies in a restriction for the tetramer spectrum, given by \(B_4^{(N)} \leq 515B_3\), which does not exist when \(B_3\) is the deepest bound trimer. This limitation, represented in Fig. 1 by the shadowed region, leads to the existence of at most three tetramers between two successive trimer states.

In case we consider \(B_3\) the deepest bound three-body state, and move this three-body reference scale related to the four-body one, such that \(B_4^{(N)}/B_3 \to \infty\) (together with \(B_2 = 0\)), an infinite tetramer levels should also appear, with level separation given by
FIGURE 3. Section of the four-boson scaling plot given in Fig. 2, limited to the region where we have recent available results. Within our scaling plot, in the upper frame we show the results of Hammer and Platter [19], in the lower left frame we have the Stecher et al. [17, 18] results, and in the lower right plot, the Deltuva results [16]. As shown, besides the fact to be limited to a small region of the plot, such results are consistent with the same scaling behavior that we are revealing with our renormalized zero-range approach. This is evidencing the role of a four-body independent scaling behavior.
$\sqrt{B_4^{(N-1)}} \approx 0.085 \sqrt{B_4^{(N)}}$. However, each other three-body excited state will lead to a limited tetramer spectrum.

Our scaling plot, which was obtained by considering a renormalized zero-range approach, is the realization of the existence of a universal behaviour represented by a new independent four-body scale. The consistency of our conclusions, on the existence of such a scale, can be verified by considering the results obtained by other authors, which have calculated tetramer spectrum using different methods. In particular, we are selecting the results obtained by Hammer and Platter [19], by Stecher et al. [17, 18], and by Deltuva [16]. The results are shown in the three panels of Fig. 3. As we can see, such results are also consistent with our conclusion on the existence of an independent four-body scale, besides the fact that they are limited to a small section of our scaling function. We believe that, within a more detailed analysis by those authors, varying the parameters of their interactions, they can also reach a scaling function similar to the one we have obtained. Range effects, not considered in our approach, can be more deeply investigated, by using different two-body potentials.

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**REFERENCES**