Due: Tuesday, October 6, 9 am

1. (a) Develop the electric potential for the array of charges shown in Arfken and Weber, Figure 12.5. You need only consider the case \( r > 2a \). (Hint: you can treat it as two dipoles.)

(b) Consider a point electric quadrupole with quadrupole moment \( Q \) that is placed at \( z = b \) and a point electric quadrupole with quadrupole moment \( -Q \) that is placed at \( z = 0 \). Let \( b \to 0 \) keeping \( Qb \) constant. Show that by a suitable choice of the value of \( Qb \) you can reproduce the potential obtained in (a) in the region where \( a \ll r \).

(c) Show that the construction from (b) means that the potential of a point octupole may be obtained by differentiating the potential of a point quadrupole. What variable should the derivative be taken with respect to?

(d) By operating in spherical polar coordinates, show that
\[
\frac{\partial}{\partial z} \left[ \frac{P_n(\cos \theta)}{r^{n+1}} \right] = -(n + 1) \frac{P_{n+1}(\cos \theta)}{r^{n+2}}.
\]

(e) What is the connection between your results of (b) and (c)? Given the result of (c) can you generalize the finding of (b)?

2. (a) Neutrons are being scattered by a nucleus of mass \( A \). In the center-of-mass system the scattering is isotropic. Boosting to the laboratory system we find that the average of the cosine of the angle of deflection of the neutron is:
\[
\langle \cos \psi \rangle = \frac{1}{2} \int_0^{\pi} \left. \frac{A \cos \theta + 1}{\sqrt{A^2 + 2A \cos \theta + 1}} \sin \theta \right| d\theta.
\]
(Note: you do not need to show that Eq. (1) is true.) Show that \( \langle \cos \psi \rangle = \frac{2}{3A} \).

(b) A function \( f(x) \) is expanded in a Legendre series:
\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x).
\]
Show that:
\[
\int_{-1}^{1} [f(x)]^2 dx = 2 \sum_{n=0}^{\infty} \frac{a_n^2}{2n + 1}.
\]
This is Parseval’s identity for the Legendre series.

3. As an extension of the example discussed in class, find the potential \( \psi(r, \theta) \) produced by the charged conducting disk of radius \( a \) shown in Fig. 12.10 of Arfken and Weber. Take the charge density on each side of the disk to be:
\[
\sigma(\rho) = \frac{q}{4\pi a \sqrt{a^2 - \rho^2}}, \quad \rho^2 = x^2 + y^2.
\]
You may assume \( r > a \).

(Hint: the integral you need to compute to find the potential along the z-axis can be rendered straightforward by a couple of astute changes of variables.)

4. Proving the Rayleigh equation

(a) Show that

\[
\int_{-1}^{1} x^n P_n(x) \, dx = \frac{2^{n+1} n! n!}{(2n+1)!}.
\]

(b) Prove that:

\[
e^{ikr \cos \gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos \gamma).
\]  

This can be done as follows:

i. Solve Eq. (2) for \( a_n j_n(kr) \).

ii. Differentiate your result \( n \) times with respect to \( (kr) \) and set \( r = 0 \) to eliminate the \( r \)-dependence.

iii. Evaluate the remaining integral using your result from (a), and so show that \( a_n = i^n (2n+1) \).

(c) Use Eq. (2) to demonstrate that:

\[
j_n(kr) = \frac{1}{2i^n} \int_{-1}^{1} e^{ikr\mu} P_n(\mu) \, d\mu.
\]

5. (a) Show that the generating function of Bessel functions obeys \( g(u+v, t) = g(u, t)g(v, t) \).

(b) Use this result and uniqueness of power series to demonstrate that

\[
J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u) J_{n-s}(v),
\]

and:

\[
J_0(u+v) = J_0(u)J_0(v) + 2 \sum_{s=1}^{\infty} J_s(u)J_{-s}(v).
\]

6. Prove that:

\[
\frac{\sin x}{x} = \int_{0}^{\pi/2} J_0(x \cos \theta) \cos \theta \, d\theta
\]
\[
\frac{1 - \cos x}{x} = \int_{0}^{\pi/2} J_1(x \cos \theta) \, d\theta.
\]

(Hint: The definite integral

\[
\int_{0}^{\pi/2} \cos^{2n+1} \theta \, d\theta = \frac{(2n)!!}{(2n+1)!!},
\]

may be useful.)
7. A function $f(x)$ is expressed as a Bessel series:

$$f(x) = \sum_{n=0}^{\infty} a_n J_m(\alpha_{mn} x). \quad (3)$$

(a) Write down the Parseval relation for the expansion (3).
(b) Use the relation of (a), and the Bessel series of the function $f(x) = x$, to show that

$$\sum_{n=1}^{\infty} \alpha_{mn}^2 = \frac{1}{4(m+1)}.$$

8. A treatment of scattering from nuclear targets in which the nucleus is thought of as a black disk of radius $R$ yields the scattering amplitude:

$$f(\theta) = -\frac{i}{\lambda} \int_{0}^{2\pi} d\phi \int_{0}^{R} \exp(i k \rho \sin(\theta) \sin(\phi)) \rho \, d\rho \, d\phi,$$

with $\theta$ the scattering angle, and $\lambda$ the de Broglie wavelength of the scattered particles. If the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

show that:

$$\frac{d\sigma}{d\Omega} = \pi R^2 \frac{1}{\pi} \left[ \frac{J_1(k R \sin \theta)}{\sin \theta} \right]^2,$$

where $k = 2\pi/\lambda$ is the wave number.

9. In a Maxwellian distribution the fraction of particle with speeds in the interval $(v, v + dv)$ is $dN$, with:

$$\frac{dN}{N} = \alpha \exp \left( -\frac{mv^2}{2kT} \right) v^2 dv,$$

where $m$ is the mass of the particles, $T$ the temperature, and $k$ is Boltzmann’s constant.

(a) Show that in order for this distribution to be properly normalized (i.e. the total number of particles to be $N$) the constant $\alpha$ must have the value:

$$\alpha = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2}.$$

(b) Then show that the average of $v^n$, defined as:

$$\langle v^n \rangle \equiv \frac{1}{N} \int v^n dN,$$

is given by:

$$\langle v^n \rangle = \left( \frac{2kT}{m} \right)^{n/2} \frac{\Gamma \left( \frac{n+3}{2} \right)}{\Gamma \left( \frac{3}{2} \right)}.$$

10. Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine their residues.