Section 3

Variational Principle $\Rightarrow$ Lagrange Equations:

Hamilton's Principle:

The true path taken by a system (defined by generalized coordinates \( q_1(t), q_2(t), \ldots, q_n(t) \)) from time \( t_1 \) to \( t_2 \) is such that

\[
I = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), \ddot{q}_i(t), \ldots, t) \, dt
\]

is stationary with respect to any infinitesimal variations from the true path from point \( \{q_1(t_1)\} \) to point \( \{q_1(t_2)\} \).

\[
I = \int_{t_1}^{t_2} L \, dt = \text{Action (or Hamilton's first principal function)}
\]

Note: The end points are "fixed," i.e., variations in path are taken to be zero at end points.

Configuration space: \( I' = \int_{t_1}^{t_2} L \, dt = I_s + SI' = I_s \) to first order path.

I' is for small deviations from the path followed by the system \( C_0 \), the action integral remains unchanged, i.e., \( SI = 0 \).

Now, we want to show that (Hamilton's Principle leads to Lagrange's Equations!)

(sometimes called the principle of least action)
Added note:

Hamilton's Principle is sometimes called the Principle of Least Action. (Because \( I = \int_{t_1}^{t_2} L dt = \text{Action} \))

But that would imply that the stationary value of \( I(q(t)) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt \) (where \( q_0(t) \) is the true path taken in configuration space) always a minimum w.r.t variations from \( q_0(t) \). This is not always the case.

Hamilton's Principle only requires \( I \) to be stationary w.r.t. variations from \( q_0 \). The value of \( I \) could be a minimum, maximum, or a saddle.

(A saddle would mean \( I \) increases from \( I_0 \) for some finite functional variations from \( q_0(t) \) and decreases for other finite functional variations from \( q_0(t) \).)

To find out what kind of stationary point \( I = I_0 \) is we must look at second order terms in the expansion of \( I \) about \( q = q_0 \).

It turns out that it is always a minimum or saddle and never is a maximum for Hamilton's Principle.

Excellent Reference:

This is a problem in the calculus of variations. Consider first a one-dimensional problem.

Wish to find the function \( y(t) \) which makes the integral

\[
I = \int_{t_1}^{t_2} F(y, y', t) \, dt
\]

stationary with respect to small changes in the function \( y(t) \) with fixed end points \( y(t_1) = y_1, y(t_2) = y_2 \).

Note \( y(t) \) defines the path over which the integral is taken.

\( F \) is "nice" function of \( y, y', \) and \( t \), i.e., continuous and differentiable.

\[
S_y(t) = S_y(t_2) = 0 \quad \text{Fixed end points}
\]

Suppose we let \( y(t) = y_s(t) + S_y(t) \) where

\( y_s(t) \) is the path that makes the integral \( I \) stationary \( (I = \min, \max, \text{or saddle wrt } S_y(t)) \) and \( S_y(t) \) is a function which represents a small deviation from the "correct" path \( y_s(t) \), \( |S_y(t)| \ll 1 \) for all \( t \).

Also \( \frac{d(S_y)}{dt} \ll 1 \) for all \( t \).

Then \( y = \frac{dy}{dt} = y_s + \frac{d(S_y)}{dt} \) and

\[
F(y, y', t) = F(y_s + S_y, y_s' + \frac{d(S_y)}{dt}, t)
\]

\[
= F(y_s, y_s', t) + \frac{\partial F}{\partial y} S_y(t) + \frac{\partial F}{\partial y'} \frac{d(S_y)}{dt}
\]

\( \text{at a fixed time } t \).
Now
\[ I = I_s + S_I = \int_{t_1}^{t_2} F \, dt = \int_{t_1}^{t_2} F(y, \dot{y}, t) \, dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \, \dot{y} \, dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{y}} \, y \, dt \]

Thus
\[ S_I = \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \, \dot{y} \, dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{y}} \, y \, dt = 0. \]

Now consider the second term on R.H.S.: (Integrate by parts)
\[ u(t) = \dot{y}(t) \quad ; \quad du = \frac{d(\dot{y})}{dt} \quad dt \]
\[ v(t) = \frac{\partial F}{\partial y} \quad ; \quad dv = \frac{d(\frac{\partial F}{\partial y})}{dt} \quad dt \]

And,
\[ \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \frac{d(\dot{y})}{dt} \, dt = \left[ uv \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} u \, dv. \]

Thus,
\[ \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \frac{d(\dot{y})}{dt} \, dt = \left. \frac{\partial F}{\partial y} \dot{y} \right|_{t_1}^{t_2} \quad - \quad \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \, \dot{y} \, dt. \]

0 since \( \dot{y}(t) = \dot{y}(t_2) = 0 \)
(Fixed end points)

Thus,
\[ S_I = \int_{t_1}^{t_2} \frac{\partial F}{\partial y} \, \dot{y} \, dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{y}} \, y \, dt = 0 \quad \text{for any} \quad \dot{y}(t). \]

This requires
\[ \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{y}} \right) - \frac{\partial F}{\partial y} = 0 \quad \text{for all} \quad t. \]

\[ \text{Euler-Lagrange differential eqn.} \]
Thus, we have shown that the function \( y(t) \)
which is such that \( y(t_1) = y_1 \) and \( y(t_2) = y_2 \)
and which makes the integral
\[
I = \int_{t_1}^{t_2} F(y, \dot{y}, t) \, dt
\]
stationary with small variations
must satisfy the Euler–Lagrange equation:
\[
\frac{d}{dt}\left( \frac{\partial F}{\partial \dot{y}} \right) - \frac{\partial F}{\partial y} = 0
\]

Example: Shortest distance between two points on the plane is a straight line.

\[
S = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2} \, dx
\]

\[
ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{1 + (\dot{y}^2)} \, dx = \sqrt{1 + y'^2} \, dx
\]

\[
S = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx
\]

\[
F = \sqrt{1 + y'^2} \quad \frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + y'^2}} \quad \frac{\partial F}{\partial y} = 0
\]

\[
\frac{d}{dt}\left( \frac{\partial F}{\partial \dot{y}} \right) = 0 \Rightarrow \frac{\partial F}{\partial \dot{y}} = \text{const.} \quad \Rightarrow \quad \dot{y} = \text{const.}
\]

\[
\frac{dy}{dx} = \text{const.}
\]

or \( y(x) = mx + b \) = equation for straight line.
The brachistochrone problem:

Find the path which minimizes the time it takes a particle starting from rest to slide without friction from point A to a lower point B.

[Answer: A cycloid with the cusp at point A.]

\[ T = \int_A^B \frac{ds}{v}, \quad (ds)^2 = (dx)^2 + (dy)^2 \]

Energy is conserved \( \Rightarrow \frac{1}{2}mv^2 = mg y \)

\[ v = \sqrt{2gy} \]

Could choose either \( x \) or \( y \) as the dependent variable.

Goldstein chooses \( y = y(x) \) but it is easier to solve the problem if we choose \( x \) as the dependent variable.

Take \( x = x(y) \Rightarrow \frac{ds}{v} = \frac{1}{\sqrt{2g}} \sqrt{\frac{x^2 + 1}{y}} dy \quad \Rightarrow \quad \frac{dx}{dy} \]

then \[ T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{x^2 + 1}{y}} dy \Rightarrow F(x, \dot{x}, y) = \sqrt{\frac{x^2 + 1}{y}} \]

Euler-Lagrange equation: \( \frac{d}{dy} \left( \frac{\partial F}{\partial \dot{x}} \right) - \frac{\partial F}{\partial x} = 0 \)

but \( \frac{\partial F}{\partial x} = 0 \)

Thus \( \frac{\partial F}{\partial x} = \text{constant} = \frac{x}{\sqrt{y(x^2 + 1)}} = C \)

Note:
This is why it is easier to choose \( x \) as the dependent variable. \( F(x, \dot{x}, y) \) does not depend on \( x \) and hence immediately have first integral.
\[
\dot{x}^2 = c^2 y \dot{x}^2 + \dot{c}^2 y
\]

\[
x = \frac{dx}{dy} = \sqrt{\frac{c^2 y}{1 - c^2 y}}
\]

Take point A as (\(x_A = 0, y_A = 0\))

\[
\int_0^x dx = \int_0^y \sqrt{\frac{c^2 y}{1 - c^2 y}} dy
\]

Let \(c^2 y = \sin^2 \theta\) \(\therefore dy = \frac{1}{c} \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta\)

\[
\sqrt{1 - c^2 y} = \sqrt{1 - \sin^2 \frac{\theta}{2}} = \cos \frac{\theta}{2}
\]

Then

\[
x = \frac{1}{c} \int_0^\theta \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta = \frac{1}{c^2} \int_0^\theta \sin^2 \frac{\theta}{2} d\theta
\]

\[
\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} = \frac{1}{2} \int_0^\theta (1 - \cos \theta) d\theta
\]

\[
\therefore x = a (\theta - \sin \theta) \quad ; \quad a = \frac{1}{2c^2}
\]

\[
y = \frac{1}{a} \sin^2 \frac{\theta}{2} = a (1 - \cos \theta)
\]

\((0, 0) A \quad a \theta \quad x \quad B (x_B, y_B) \quad \theta \quad a \quad y_B \quad 2a \quad a = \frac{y_B}{2}
\]

Note: Find a in terms of \(x_B\) and \(y_B\):

\[
a \theta_B = x_B \Rightarrow \sin \theta_B = 0
\]

\[
\therefore \theta_B = \pi \Rightarrow \cos \theta_B = -1
\]

\[
y_B = 2a \quad \text{or} \quad a = \frac{y_B}{2}
\]
Now back to Lagrange Equations:

Action \( J = \int_{t_1}^{t_2} L(q_1, \dot{q}_1, \ldots, q_n, \dot{q}_n, t) \, dt \)

where \( L = T(q_1, q_2, \ldots, q_n) - U(q_1, q_2, \ldots, q_n, t) \equiv \text{Lagrangian}. \)

Now Hamilton's principle \( \delta J = 0 \) for small arbitrary deviations from path \( \{ \delta q(t) \} \) i.e.,

\[ q_1(t) = q_{1s}(t) + \delta q_1(t) \]
\[ q_2(t) = q_{2s}(t) + \delta q_2(t) \]
\[ q_n(t) = q_{ns}(t) + \delta q_n(t) \]

with fixed end points: \( \delta q(t_1) = \delta q(t_2) = 0. \)

Following the same procedure as followed for the one-dimensional case (pg 3-2 - 3-3) for each variable \( q_2(t) \) we obtain

\[ \delta J = \int_{t_1}^{t_2} \sum \delta q_k \left[ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) \right] dt = 0 \]

But all \( \delta q \) are independent and hence we must have

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) = \frac{\partial L}{\partial q_2} \]

Thus, we have proven that Lagrange equations of motion follow from Hamilton's principle if forces can be described by generalized potential and holonomic constraints.
Can generalize Hamilton's principle to include
\( \rightarrow (i) \) forces not represented by a generalized potential
and
\( \rightarrow (ii) \) non-holonomic differential constraints.

First consider forces not represented by generalized potential:

Generalization(i):

We obtained from D'Alembert's principle: (see pg 2-16)

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = \dot{Q}_k
\]
where \( Q_k = \) generalized force associated with \( q_k \).

\((\text{virtual work: } \delta W_e = Q_k \delta q_k)\)

But following the methods of calculus of variations we

have

\[
\mathcal{S} \left[ \int_{t_1}^{t_2} T \, dt \right] = \left[ \int_{t_1}^{t_2} \sum_k \left( \frac{\partial T}{\partial \dot{q}_k} - \frac{d}{dt} \left( \frac{\partial T}{\partial q_k} \right) \right) \dot{q}_k \, dt \right]
\]

\[
= - \int_{t_1}^{t_2} \sum_k Q_k \dot{q}_k \, dt
\]

Split \( Q_k \) into part represented by generalized potential and part not:

\[
Q_k = Q^g_k + Q^{ng}_k \quad \text{where} \quad Q^g_k = -\frac{\partial U}{\partial q_k} + \frac{1}{2} \frac{\partial \left( \frac{\partial U}{\partial q_k} \right)}{\partial \dot{q}_k}
\]

Then "Hamilton's principle" becomes

\[
\mathcal{S} \left[ \int_{t_1}^{t_2} L \, dt \right] = -\int_{t_1}^{t_2} Q^{ng}_k \dot{q}_k \, dt = -\text{integral of virtual work done by non-generalized potential forces.}
\]

and

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q^{ng}_k
\]
Generalization 16: (Suppose all forces are described by generalized potential but.)

Now consider non-holonomic constraints of the form \((m \text{ constraint equations})\):

\[
\sum_{\ell} a_{\alpha \ell} dq_{\ell} + a_{\alpha \ell} \frac{dq_{\ell}}{dt} dt = 0, \text{ for } \alpha = 1, 2, \ldots, m.
\]

In general \(q_{\ell}(q_0, t)\)

Then the correct equations of motion may be obtained using the method of Lagrange multipliers and methods from the calculus of variations if we assume that the path variation \(q_{\ell}(t) = q_{\ell}(t) + \delta q_{\ell}(t)\) is obtained by starting with \(q_{\ell}(t)\) and then changing \(q_{\ell}\) by amount \(\delta q_{\ell}(t)\) for each \(t\) consistent with the constraint equations but at fixed \(t\), i.e., the \(\delta q_{\ell}(t)\) are virtual displacements.

Then

\[
\sum_{\ell} a_{\alpha \ell} \delta q_{\ell} = 0 \quad \text{for } \alpha = 1, 2, \ldots, m.
\]

and introduce Lagrange undetermined multipliers \(\lambda_{\alpha}(q_0, t)\):

\[
\lambda_{\alpha} \sum_{\ell} a_{\alpha \ell} \delta q_{\ell} = 0; \quad \alpha = 1, 2, \ldots, m.
\]

Using the methods of calculus of variations

\[
\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_{\ell} \left[ \frac{\partial L}{\partial q_{\ell}} \delta q_{\ell} + \frac{\partial L}{\partial \dot{q}_{\ell}} \delta \dot{q}_{\ell} \right] dt = 0 \quad \text{by Hamilton's principle}
\]

but now the \(\delta q_{\ell}\)'s are not all independent.

Can treat them as independent if odd in the Lagrange multipliers:
\[
\int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k + \sum_{\alpha} \lambda_\alpha \delta q_k \right] dt = 0
\]

Now \( \delta q_k \) may be treated as independent:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_{\alpha} \lambda_\alpha \delta q_k \quad k = 1, 2, \ldots, 3N \]

together with the \( m \) equations of constraint:

\[
\alpha \dot{q}_\alpha + \alpha \ddot{q}_\alpha = 0 \quad \alpha = 1, 2, \ldots, m
\]

gives \( 3N + m \) equations for the \( 3N + m \) unknowns:

\( q_1(t), q_2(t), \ldots, q_{3N}(t) \) and \( \lambda_1, \ldots, \lambda_m \)

**Physical meaning of the \( \lambda_\alpha \):**

Notice that we also (as a bonus for the extra work) determine in this case the forces of constraint because, if we were to ignore the constraint equations and replace their effect by forces then we would have a situation as described on pg. 3-8. The forces of constraint would be non-generalized potential forces and the equations would be of the form given at bottom of pg. 3-8 with the \( \mathbf{Q}^c = \) force of constraint.

Thus, the force of constraint associated with coordinate \( q_k \) is given by:

\[
\mathbf{Q}^c_k = \sum_{\alpha=1}^{m} \lambda_\alpha \dot{q}_k
\]
Using Lagrange multiplier method.

Example: 2-D motion \( \Rightarrow \) Coord. \((r, \theta)\)

Particle sliding in smooth circular bowl: 2-D case

Constraint: \( r = \text{constant} = a \Rightarrow \ \dot{r} = 0 \)

\[
T = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right)
\]

\[
V = -mg r \cos \theta
\]

\[
L = T - V = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) + mg r \cos \theta
\]

\[
\frac{\partial (T - V)}{\partial \bar{q}} - \frac{\partial \lambda}{\partial \bar{q}_{\lambda}} = \sum \lambda \frac{\partial \bar{q}}{\partial \bar{q}_{\lambda}}
\]

Here we have only one constraint equation.

so \( \lambda = 1 \) only, \( \Rightarrow \) one \( \lambda \),

and \( \lambda \bar{q}^3 = (r, \theta) \)

\[
r: \quad \frac{\partial \lambda}{\partial r} = m \ddot{r} + m r \dot{\theta}^2 + mg \cos \theta
\]

or

\[
\ddot{r} = m r \dot{\theta}^2 - mg \cos \theta = \lambda \tag{1}
\]

\[
\theta: \quad \frac{\partial \lambda}{\partial \theta} = m r^2 \ddot{\theta} - mg r \sin \theta
\]

or

\[
2mr \dddot{\theta} + m r^2 \ddot{\theta} + mg r \sin \theta = 0 \tag{2}
\]

And constraint equation:

\[
\dot{r} = 0 \tag{3}
\]

Putting \(3) \Rightarrow \dddot{r} = 0 \) into \(1\) gives \( \lambda = -(m r^2 \dot{\theta}^2 + mg \cos \theta) \)

Thus, constraint force associated with \( r: \)

\[
Q_r = \lambda \cdot a_r = \lambda
\]

\[

dW_r = Q_r \bar{r} \Rightarrow \bar{Q_r} = \text{Normal force of constraint along } r
\]

\[
\therefore N = -(m r^2 \dot{\theta}^2 + mg \cos \theta) \bar{r}
\]

due to gravity
Goldstein Problem as an example:

An interesting example using Lagrange multipliers:

Mass \( m \) slides without friction on an inverted sphere. It starts from rest at top; at what angle will the mass fall off the sphere?

Method: Find force of constraint associated with coord. \( r \) and then find the angle \( \theta \) at which \( Q_r = 0 \).

Coord: \((r, \theta)\)  

\( r, \dot{r} - a_r, \theta, \dot{\theta}, a_\theta = 0 \)

Constraint eqn: \( r = a \Rightarrow \dot{r} = 0 \Rightarrow a_r = 1, a_\theta = a_t = 0. \)

\[ T = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) \]
\[ V = mg r \cos \theta \]

\[ L = T - V = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - mg r \cos \theta. \]

\[ r: \quad \frac{2 \dot{r}}{\dot{r}} = -mr^2 \dot{\theta} - mg \cos \theta; \quad \lambda a_r = \lambda \]

\[ \therefore \quad \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda a_r \Rightarrow \quad \dot{r}^2 + m r^2 \dot{\theta}^2 - mg r \cos \theta = \lambda \quad \text{(1)} \]

\[ \theta: \quad \frac{\partial L}{\partial \dot{\theta}} = m r^2 \ddot{\theta} \quad ; \quad \frac{\partial L}{\partial \theta} = m g \sin \theta \quad ; \quad \lambda a_\theta = 0 \]

\[ \therefore \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \dot{\theta}} = \lambda a_\theta \Rightarrow \quad m r^2 \ddot{\theta} - mg r \sin \theta = 0 \quad \text{(2)} \]

Constraint equations: \[ \dot{r} = 0 \]

\[ \dot{\theta} = 0 \]
From the constraint equation: \( \ddot{r} = 0 \Rightarrow \ddot{r} = 0 \quad r = a \)

Put this into (4):

\[
\lambda = mg \cos \Theta - ma \dot{\Theta}^2
\]

Now the force normal to the surface is \( Q_r \) (Since \( dW = N \cdot dr = Q_r \cdot dr \))

\[
Q_r = \lambda a \dot{\Theta} = \lambda = mg \cos \Theta - ma \dot{\Theta}^2 \quad \text{(from (4))}
\]

1. Condition \( Q_r = 0 \) is same as condition \( \lambda = 0 \)

or when \( \Theta \) is such that

\[
mg \cos \Theta - ma \dot{\Theta}^2 = 0 \quad \text{(5)}
\]

We have a conservative system with holonomic constraints so that total energy is conserved:

\[
T + V = \text{const. of motion}
\]

\[
\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\Theta}^2) + mg r \cos \Theta = \text{constant}
\]

\[
\dot{\Theta}^2 = -\frac{mg \cos \Theta}{m a^2} + K = K - \frac{2 \pi}{a} \cos \Theta
\]

Initial conditions: \( \dot{\Theta} = 0 \) when \( \Theta = 0 \Rightarrow K = \frac{2 \pi}{a} \)

\[
\dot{\Theta}^2 = \frac{2 \pi}{a} (1 - \cos \Theta)
\]

Put (3) into (5) and solve for \( \Theta \):

\[
mg \cos \Theta - ma \frac{2 \pi}{a} + ma \frac{2 \pi}{a} \cos \Theta = 0
\]

\[
3 \cos \Theta = 2 \quad \Rightarrow \quad \cos \Theta = \frac{2}{3} \quad \text{(indep. of } a)\]
Symmetry and Mathematical Properties of the Lagrangian:

I) Non-interacting systems: \( L = l_A + l_B \)

\[ A \quad B \]

\[ A \text{ and } B \text{ do not interact} \]

\[ \int_{t_1}^{t_2} l_A \, dt = 0 \quad \text{and} \quad \int_{t_1}^{t_2} l_B \, dt = 0 \]

Thus \( \int_{t_1}^{t_2} (l_A + l_B) \, dt = 0 \)

and \( L = l_A + l_B \) is a suitable Lagrangian for system \( A+B \).

II) Scalar invariance: \( L' = \alpha L \), \( \alpha = \text{constant} \neq 0 \).

If \( L \) is a valid Lagrangian then so is \( L' \).

Clearly \( \int_{t_1}^{t_2} l' \, dt = \alpha \int_{t_1}^{t_2} l \, dt = 0 \Rightarrow \) same equations of motion.

III) Gauge Invariance: \( L' = L + \frac{d}{dt} f(q, \dot{q}, t) \)

If \( L \) is valid Lagrangian, then so is \( L' \).

Proof:

\[ \int_{t_1}^{t_2} (L' \, dt) = \int_{t_1}^{t_2} L \, dt + \int_{t_1}^{t_2} \left( \frac{d}{dt} f(q, \dot{q}, t) \right) dt \]

\[ = 0 \]

\[ \int_{t_1}^{t_2} \left( f(q, \dot{q}, t) \right) dt \]

\[ = 0 \text{ since } \int \text{ keeps end points fixed} \]

Note: II and III show that the Lagrangian for a given system is not unique!
IV. Generalized Translational invariance $\Rightarrow$ conservation of generalized momenta

Where are the conservation laws which are so useful in Newtonian Mechanics?

For example: In Newtonian mechanics, if $V$ does not contain the coordinate $x$ then $F_x = \frac{\partial V}{\partial x} = 0$ and the linear momentum along $x$-direction: $m\dot{x}$ must be constant.

When constraints are present, the situation will generally be more complicated because of the forces of constraints. However, when using independent generalized coordinates similar conservation laws can be found again in terms of only the "applied" forces (i.e., the constraint forces are not explicitly considered).

Define generalized momentum $p_x$, conjugate to generalized coordinate $q_x$ is defined:

$$p_x = \frac{\partial L}{\partial \dot{q}_x}$$

$p_x$ is called canonical momentum or conjugate momentum.

Note: If have free particles: $T = \frac{1}{2}m(x^2 + y^2 + z^2)$, $V = 0$

and $p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$ = ordinary linear momentum
However, for a particle in a magnetic field:

\[ L = T - U \]

\[ U = g \phi (x,y,z,t) - q A \cdot V = g \phi - q (A_x \dot{x} + A_y \dot{y} + A_z \dot{z}) \]

Thus

\[ \dot{p} = \frac{\partial T}{\partial x} - \frac{\partial U}{\partial x} = m \dot{x} + q A_x \]

*extra term!*

Define: cyclic coordinate \( q_j \):

\[ L(q_1, q_2, ..., q_j, ..., q_l, t) \]

If \( L(q_1, q_2, ..., q_j, ..., q_l, t) \) does not contain the coordinate \( q_j \), then \( q_j \) is called cyclic.

Note: \( L \) may contain \( q_j \) even though it does not contain \( q_j \).

Conservation of conjugate momentum:

If \( q_j \) is a cyclic coordinate then

the corresponding conjugate momentum, \( p_j = \) constant.

Proof: Follows directly from Lagrange's equations:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} = 0 \text{ for cyclic } q_j \]

\[ \therefore \frac{d}{dt} p_j = 0 \text{ if } q_j \text{ is cyclic.} \]
Connection between symmetry properties of the system and conservation of canonical momenta:

If a particular coordinate, \( q_j \), is cyclic it means that any arbitrary translation of the system by replacing \( q_j \) with \( q_j + x \) (all other \( q_i \)'s remain fixed) has no effect. The system has a symmetry property such that it is invariant under an arbitrary translation.

The canonical momentum conjugate to the invariant translation coordinate is conserved.

Note: If the cyclic coordinate is a rotation then the system is invariant under rotation about a given axis. And the corresponding canonical momentum will be an angular momentum of the system about the given axis.

Angular momentum:

Spherically symmetric system \( \Rightarrow L_\phi, L_\theta, L_z \) all constant
Axial symmetry (about \( z \) axis, say) \( \Rightarrow L_z \) constant

The above are special cases of a more general theorem relating symmetry properties of a system to constants of motion (see pg. 3-21).
Well, what about conservation of Energy?!

It turns out that energy conservation is a special case of a more general conservation law:

IV) If \( L \) does not depend explicitly on time, \( t \)

\( L(\mathbf{q}, \dot{\mathbf{q}}, t) \)

i.e. \( L(\mathbf{q}, \dot{\mathbf{q}}) \) then the function

\[
\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{k} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = \text{Hamiltonian (Jacobi's integral)}
\]

is conserved.

If \( L \) does not depend explicitly on time, then the zero of time is arbitrary.

\[ \text{Note: } L \text{ is usually not a constant of the motion even though it may not depend on } t \text{ explicitly.} \]

Proof:\n
\[
\frac{d}{dt} L = \sum_{k} \frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}
\]

Using Lagrange's eqns \( \frac{\partial L}{\partial \dot{q}_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \)

\[
\frac{d}{dt} L = \sum_{k} \left( \frac{\partial L}{\partial q_k} \ddot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \dot{\ddot{q}_k} + \frac{\partial L}{\partial t} \right)
\]

\[
= \frac{d}{dt} \left( \sum_{k} \frac{\partial L}{\partial q_k} \dot{q}_k \right) + \frac{\partial L}{\partial t}
\]

or

\[
\frac{d}{dt} \left[ \sum_{k} \frac{\partial L}{\partial q_k} \dot{q}_k - L \right] = -\frac{\partial L}{\partial t}
\]

Note that \( \mathcal{H} \) is not always the total mechanical energy \( T + V \) of the system.
When does \( H = T + V \) = total mechanical energy?

\[
H = \sum \frac{\dot{q}_i}{2} \frac{\partial \dot{q}_i}{\partial q_j} - L, \quad \text{?}
\]

**Answer:** \( H = T + V \) if

(a) conservative velocity independent forces, \( \phi \) indp.

\[ F = - \nabla V \quad \Rightarrow \quad V = V(\mathbf{q}_3) \]

and

(b) time independent constraint equations. (holonomic)

\[ \Pi = \Pi(\mathbf{q}_3) \Rightarrow \text{no explicit time dep.} \]

\( \Rightarrow \) \( T \) is homogeneous of degree 2 in \( \mathbf{q}_3 \)

**Proof:** Given the above conditions (a) and (b)

\[
T = \frac{1}{2} \sum m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i
\]

Transform to generalized coordinates:

\[
\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}_3)
\]

\[
\dot{q}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_3}
\]

\[
\dot{\mathbf{r}}_i = \sum \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j
\]

\[
T = T(\mathbf{q}_3, \dot{\mathbf{q}}_3) = \frac{1}{2} \sum \sum f_{kl} \dot{q}_k \dot{q}_l
\]

Thus, \( T \) is of degree 2 in the variables \( \mathbf{q}_3, \dot{\mathbf{q}}_3 \).

\[
L = T(\mathbf{q}_3, \dot{\mathbf{q}}_3) - V(\mathbf{q}_3)
\]

\[
\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left( \sum \sum f_{kl} \dot{q}_k \dot{q}_l \right) = 2 \sum f_{jk} \dot{q}_j \dot{q}_k
\]

And

\[
\sum \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = \sum \sum 2 f_{jk} \dot{q}_j \dot{q}_k = 2T
\]

\[
H = \sum \frac{\dot{q}_j}{2} \frac{\partial \dot{q}_j}{\partial q_j} - L = 2T - (T - V) = T + V
\]
Note: The questions of whether

\[ h \neq T + V \]

\[ h \neq \text{const.} \]

\[ T + V \neq \text{const.} \]

are different questions.

It can be that \( h \neq T + V \) but \( h = \text{const.} \),
and it also can be that \( h \neq \text{const.} \) and
\( h \neq T + V = \text{const.} \).

For example consider a conservative system in a
cartesian coord. system. Now generalized coordinates
could be chosen which give the positions of all particles
relative to a moving coord. system (for example, one
with center oscillating like \( A \sin \omega t \)). \( h \) would
no longer = \( T + V \) and \( h \) would not be conserved
but \( E = T + V \) would, of course, still be conserved.
Noether's Theorem (1918)  

During our discussion of "cyclic" coordinates it was pointed out that this was the result of a translational symmetry in configuration space, $(q, \dot{q})$-space.

There is a general theorem by Nöther stating that every symmetry in configuration space is associated with a corresponding constant of the motion.

Consider a family of transformations from gas. coord. $\{q_3\}$ to gas. coord. $\{\eta_3\}$. The various transformations in the family are distinguished by a single continuous parameter $\varepsilon$.

\[ \eta_3 = \eta_3(\{q_3\}, \varepsilon) = \eta_3^\varepsilon(\eta_3) \]

$\varepsilon$ is a parameter in the transformation.

And there must also be the inverse transformation

\[ q_3 = q_3^\varepsilon(\eta_3) \]

Also suppose $\eta_3 = q_3$ when $\varepsilon = 0$. 

---

The greatest of women mathematicians.
Note that it is enough to give \( \dot{\eta}_e(Q) \) since \( \dot{\eta}_e \) can be determined from \( \eta_e^E(Q) \):

\[
\dot{\eta}_e = \frac{\partial \eta_e^E}{\partial \dot{\eta}_x} \dot{\eta}_x = \eta_e^E(2, \dot{2})
\]

Similarly,

\[
\dot{\eta}_x = \frac{\partial \eta_e^E}{\partial \dot{\eta}_x} \dot{\eta}_x = \eta_e^E(\eta, \dot{\eta})
\]

This type of transformation is called a \( \eta \)-transformation or a "point" transformation.

Now consider how \( h \) depends on these transformations:

Given \( L(\eta, \dot{\eta}, t) \), then

\[
L'(\eta, \dot{\eta}, t; \epsilon) = L(\eta^E(\eta), \dot{\eta}^E(\eta, \dot{\eta}), t)
\]

since at \( \epsilon = 0 \) \( q_x = \eta_x \) it follows that \( \dot{q}_x = \dot{\eta}_x \) at \( \epsilon = 0 \) and thus

\[
L(\eta, \dot{\eta}, t; \epsilon = 0) = L(\eta, \dot{\eta}, t)
\]

But in general \( S_e L \equiv \frac{\partial L}{\partial \dot{\eta}_e} \bigg|_{\epsilon = 0} \neq 0 \) for arbitrary \( \epsilon \).

We say \( L \) is invariant under a particular \( \epsilon \)-family of \( \eta \)-transformations if \( L \) is indep. of \( \epsilon \) at \( \epsilon = 0 \), i.e.

\[
\left. \frac{\partial L}{\partial \dot{\eta}_e} \right|_{\epsilon = 0} = 0 \quad \text{for arbitrary } \dot{\eta}_e
\]

Thus \( L \) is invariant if

\[
\left. \frac{\partial L}{\partial \dot{\eta}_e} \right|_{\epsilon = 0} = 0 \quad \text{for all } (\eta, \dot{\eta}) = (q, \dot{q})
\]

If this is so, then the particular transformation is said to be associated with a \( \eta \)-symmetry of the Lagrangian.
Evaluating $\mathcal{L}^e$: $\mathcal{L} = \mathcal{L}(q^e(\eta), \dot{q}^e(\eta, \dot{\eta}), t)$

$$\mathcal{L}^e = \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} = \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} + \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right) \frac{d\dot{q}^e}{d\varepsilon} \quad \text{(sum over } \alpha)$$

Note: $\frac{d\varepsilon}{d\varepsilon} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^e} = -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon}$ (since $\varepsilon$ is taken to be held const. while taking the time derivative).

Thus $\frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} = \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \left[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right] = \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right] - \left[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \right] \frac{d\dot{q}^e}{d\varepsilon}$

Thus $\mathcal{L}^e = \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} = \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right) \right] \frac{d\dot{q}^e}{d\varepsilon} + \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right] \frac{d\dot{q}^e}{d\varepsilon}$

$$\mathcal{L}^e = \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \bigg|_{\varepsilon=0} = 0 \Rightarrow \frac{d\dot{q}^e}{d\varepsilon} \bigg|_{\varepsilon=0} = 0 \Rightarrow \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \right] \bigg|_{\varepsilon=0} = 0 \bigg|_{\varepsilon=0}$$

Therefore, if $\mathcal{L}^e \bigg|_{\varepsilon=0} = 0$, we have a constant of the motion:

$$K(t, \dot{t}, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}^e} \frac{d\dot{q}^e}{d\varepsilon} \bigg|_{\varepsilon=0} \quad \text{(sum over } \alpha)$$
Examples of Noether's Theorem:

To find the constant of motion carry out the differentiation:

$$\frac{\partial L}{\partial \dot{q}_\lambda} \quad \text{and} \quad \frac{\partial \mathcal{E}}{\partial \epsilon} \bigg|_{\epsilon=0}$$

this will depend on the particular family of $\varepsilon$-trans.

Cyclic coordinate case: $L(q_1, \ldots, q_n, \dot{q}_\lambda, \varepsilon)$ ($q_\lambda$ is not in $L$)

Then the appropriate transformation is:

$$\eta_\lambda = q_\lambda + \epsilon$$
$$\eta_{\lambda \neq \lambda} = q_{\lambda \neq \lambda}$$

From these transform, eqns. we see that

$$\dot{\eta}_\lambda = \dot{q}_\lambda \quad \text{for all } \lambda$$

$$q_\lambda^\varepsilon = \eta_\lambda - \epsilon \quad \text{and} \quad \frac{\partial q_\lambda^\varepsilon}{\partial \epsilon} = (-1) \delta_{\alpha \lambda} = \begin{cases} -1 & \text{if } \alpha = \lambda \\ 0 & \text{if } \alpha \neq \lambda \end{cases}$$

$$q_{\lambda \neq \lambda}^\varepsilon = \eta_{\lambda \neq \lambda}$$

so the constant of motion is

$$\mathcal{K} = \frac{\partial L}{\partial \dot{q}_\lambda} = -\mathcal{P}_\lambda$$
Easy extension of our formulation of Noether's Theorem:

\[ L'(\dot{q}, \ddot{q}, \dddot{q}, t) = L(q, \dot{q}, \ddot{q}, t) + \frac{d}{dt} f(\dddot{q}, t) \]

gives

the same Lagrange eqns (Gauge Invariance) as

the original Lagrangian \( L(q, \dot{q}, t) \), where \( f \) is any
differentiable function of \( \dddot{q} \) and \( t \).

Thus, \( L'(q, \dot{q}, \ddot{q}, t; \epsilon) = L(q, \dot{q}, \ddot{q}, t) + \frac{d}{dt} f(q, \dot{q}, \ddot{q}, \epsilon, t) \) is said to
be gauge invariant under an \( \epsilon \)-family of
\( q \)-transformations if

\[
\delta L = \frac{\partial L}{\partial \dot{q}} \frac{d\dot{q}}{d\epsilon} - \frac{\partial L}{\partial q} \frac{dq}{d\epsilon} |_{\epsilon=0}
\]

Then we can make the original \( L \) invariant
by starting with the Lagrangian \( L' = \frac{d}{dt} f(q, \dot{q}, \ddot{q}, t) \).

We have a new \( L'_{\text{New}} = L'(q, \dot{q}, \ddot{q}, t) - \frac{d}{dt} f(q, \dot{q}, \ddot{q}, t) \)

\[
\frac{\partial L'_{\text{New}}}{\partial \epsilon} = 0 = \frac{d}{dt} \left[ \frac{\partial L'_{\text{New}}}{\partial \dot{q}} \right]_{\epsilon=0} - \frac{d}{dt} f(q, \dot{q}, \ddot{q}, \epsilon, t) |_{\epsilon=0}
\]

Thus, we have the const. of motion

\[
K(q, \dot{q}, \ddot{q}, t) = \left( \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \epsilon} - \frac{\partial L}{\partial q} \frac{dq}{d\epsilon} \right) |_{\epsilon=0}
\]

Trivial Example:

\( L = \frac{x^2}{2} + ax \)

Let \( x' = x + \epsilon \rightarrow x = x - \epsilon \) and \( \dot{x} = \dot{x}' \)

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \dot{x}' - a = 0
\]

or \( x' = a \)

\( \therefore \dot{x} = \dot{x}' \)

Thus, \( x + at = \text{const.} \)

\[
K = \frac{\partial L}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \epsilon} - \frac{\partial L}{\partial x} \frac{dx}{d\epsilon} |_{\epsilon=0} = \dot{x} + at = \text{const.}
\]