Problem 4-1: [10 pts.] A general surface of revolution may be described in cylindrical polar coordinates \((r, \phi, z)\) by the function \(r = r(z)\). The function \(r(z)\) and its derivative \(dr/dz \equiv r'(z)\) are given. Use variational calculus to find the equation for the curve that is the shortest path between two points on this surface.

(a) Show clearly that the expression for the differential path length on the surface that is the result of displacement \(dz\) and \(d\phi\) is given by \((ds)^2 = (1 + r'^2)(dz)^2 + r^2(d\phi)^2\).

(b) Take \(z\) as the independent variable and show that the curve, \(\phi(z)\), that is the shortest path between two points on the surface is given by

\[
\phi(z) = \phi_0 + k \int_{z_0}^{z} \frac{\sqrt{r^2(z')} + 1}{r(z') \sqrt{r^2(z') - k^2}} \, dz'.
\]

(c) Thus, show that the shortest path between two points on the surface of a cylinder of radius \(a\) is a section of a helix.

(d) [Bonus 5 pts.] Clearly show that the above procedure leads to an infinite number of distinct helices that all pass through the same two points. They can’t all be the shortest path. Clearly explain how it is that all of these helices do satisfy the above criteria and how you would pick shortest path from the countably infinite set of candidates.

Problem 4-2: [10 pts.] Goldstein, Poole, and Safko, 3rd ed., Problem 24, Chapter 2, pg. 68.

Problem 4-3: [20 pts.] Goldstein, Poole, and Safko, (3rd ed.), Problem 19, Chapter 2, pg. 67.

The purpose of this problem is to show the power of simply using the form of the Lagrangian function to find constants of motion.

Note on distribution of points and hints:
[Total 10 pts.] for parts (a) - (f). For these six situations you should choose an appropriate coordinate system and argue that certain variables will not appear in the potential \(V\) affecting the symmetry of \(L\). Indicate exactly how the particular functional form of \(L\) leads to a particular dynamic function to be constant (or not) in each case.

[10 pts.] for part (g). Suppose that the helical wire of mass is given by the equations (in cylindrical polar coordinates) \((r, \phi, z)\): \(r = b; \phi - kz = 0\). Use this symmetry to show that the Lagrangian \(L\) will be invariant under the family \(q\)-transformations: \(r' = r; \phi' = \phi + k\epsilon; z' = z + \epsilon\). And then use Nöther’s theorem to find the corresponding constant of motion.
shortest path on surface of revolution problem: soln

Cylindrical polar coords \((r, \phi, z)\).

(a) \[ (ds)^2 = (dr)^2 + r^2(\phi)^2 + (dz)^2 \]

Surface of revolution given by \(r(z)\).

Given: \(r(z)\) and \(\frac{dr}{dz} = r'(z)\)

If on the surface and move \(dz\) and \(d\phi\) then
\[ dr = r' \, dz. \text{ Thus} \]
\[ (ds)^2 = r'^2(z)^2 \, (dz)^2 + r^2(\phi)^2 \, (dz)^2 = (1 + r^2(z))(dz)^2 + r^2(\phi)^2 \]

(b) Take \(z\) to be the indep. variable.

A point on the surface is \((r(z), \phi(z), z)\)

Given

A particular curve on the surface of revolution is determined by the function \(\phi(z)\) and the distance traveled in going from \((z_0, \phi(z_0))\) to \((z, \phi(z))\) is

\[ D(\phi(z)) = \int_{z_0}^{z} ds = \int_{z_0}^{z} \frac{\sqrt{(1 + r^2(z)) + r^2(\phi)^2}}{dz} \, dz \]

\[ D(\phi(z)) = \int_{z_0}^{z} \sqrt{(1 + r^2(z)) + r^2(\phi)^2} \, dz = \int_{z_0}^{z} F(\phi, \frac{d\phi}{dz}, z) \, dz \]

where \( F(\phi, \frac{d\phi}{dz}, z) = \left( (1 + r^2(z)) + r^2(\phi)^2 \right)^{\frac{1}{2}} \)

\( \frac{d\phi}{dz} \)
shortest path on surface of revolution — sol’r cent.

From variational calculus, \( P(\phi(z)) = \int_{z_0}^{z} L(\phi, \phi', z) \, dz \)

is an extremum w.r.t. small changes in the function \( \phi(z) \) when \( \phi(z) \) satisfies Euler's eq.

\[
\frac{d}{dz} \left( \frac{\partial F}{\partial \phi'} \right) - \frac{\partial F}{\partial \phi} = 0
\]

\[
F(\phi, \phi', z) = \sqrt{1 + \left( \frac{dz}{dr} \right)^2 + r^2 \phi'^2}
\]

\[
\frac{\partial F}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{d}{dz} \frac{\partial F}{\partial \phi'} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial \phi'} = k = \text{constant}
\]

\[
\frac{\partial F}{\partial \phi'} = \frac{r^2 \phi'}{\sqrt{1 + \left( \frac{dz}{dr} \right)^2 + r^2 \phi'^2}} = k \quad \text{or} \quad r^2 \phi'^2 = k \left( 1 + \left( \frac{dz}{dr} \right)^2 + r^2 \phi'^2 \right)
\]

\[
\phi = \frac{d\phi}{dz} = \frac{k \left( 1 + r'(z)^2 \right)}{r'(z) \left( r^2 - k \right)} \quad r'(z) \text{ and } r'' \text{ given}
\]

RHS is a known function of \( z \), so can integrate to get

\[
\phi(z) = \phi_0 + k \int_{z_0}^{z} \frac{r \left( 1 + r'(z)^2 \right)}{r'(z) \left( r^2 - k \right)} \, dz
\]

\( z' \) is dummy variable

\[
\phi(z) = \phi_0 + k \int_{z_0}^{z} \frac{r}{r'(z)^2 \left( r^2 - k \right)} \, dz
\]

(C) If \( r(z) = c = \text{const} \), then \( r'(z) = \frac{dr}{dz} = 0 \) and

\[
\phi(z) = \phi_0 + \frac{k}{a^2 (a^2 - k)} (z - z_0)
\]

or

\[
\phi(z) = \phi_0 + k (z - z_0) \quad \Rightarrow \quad \frac{d\phi}{dz} = k = \frac{k}{a^2 (a^2 - k)} = \text{constant}
\]

This is the equation for a helix in cylindrical polar coords.

\[
\frac{1}{k} \text{ is the "pitch" in rise length \text{ radian}}
\]
Shortest path on surface of revolution — solution cont.

(d) Bonus question — solution.

Given \( P_0 = (\phi_0, z_0) \) and \( P_1 = (\phi_1, z_1) \) we find

\[
K = \frac{(\phi_1 - \phi_0)}{(z_1 - z_0)}
\]

But, we arrive at the same point \( \phi_1 \) if

\[
(\phi_1 - \phi_0) = (\phi - \phi_0) \mod 2\pi \pm n \cdot 2\pi
\]

where \( n \) is any integer.

Thus,

\[
K = \frac{(\phi_1 - \phi_0) \mod 2\pi}{(z_1 - z_0)} \pm n \cdot \frac{2\pi}{(z_1 - z_0)}
\]

Have an infinite no. of helices that pass through \( P_0 \) & \( P_1 \).

The different \( K \) correspond to helices that wrap around the cylinder a different number of times before it get from \( P_0 \) to \( P_2 \).

Can also wrap around the cyl. in the opposite direction i.e., if \( K < 0 \) then get positive \( z_1 - z_0 \) by making \( \phi_1 - \phi_0 < 0 \).

We get all these helices as solutions to Euler's eq. since they are all local extrema in the path length. Actually they are always a local minima in the path length w.r.t. small perturbations in the path. Each helix is the minimum path with a fixed number of wraps in a particular direction.

The shortest distance is on the helix, with \( -\frac{\pi}{(z_1 - z_0)} < K \leq \frac{\pi}{(z_1 - z_0)} \).
GPS prob. 24 Ch. 2. Solution.

Given

\[ L = \frac{1}{2} m x^2 - \frac{1}{2} k x^2. \]

We know \( V(x) = \frac{1}{2} k x^2 \), thus have one dim. conservative system with a single minimum \( \Rightarrow \) oscillatory motion for any finite \( E \). Thus, the \( x(t) \) can always be expressed as a Fourier series

\[ x(t) = \sum_{j=0}^{\infty} a_j \cos(j \omega t) \quad \text{(taking } t=0 \text{ at a turning point where } x=0) \]

Now, use Hamilton's Principle to find the \( a_j \)'s and \( \omega \).

\[
I(a_0, \omega) = \int L \, dt = \int \left[ \sum_{j=0}^{\infty} \frac{m}{2} a_j \dot{x}^2 + \frac{k}{2} x^2 \sin(j \omega t) \sin(j \omega t) \right] \, dt
\]

\[
I(a_0, \omega) = \sum_{j=0}^{\infty} \int a_j \omega^2 \left[ \sin(j \omega t) \sin(j \omega t) dt - \frac{k}{2} \right] \left[ \cos(j \omega t) \cos(j \omega t) dt \right]
\]

or

\[
I(a_0, \omega) = \sum_{j=0}^{\infty} a_j \omega^2 \left[ \frac{2 \pi}{j \omega} \left( \frac{1}{2} \delta_{j,0} + \frac{1}{2} \delta_{j,k} \right) \right]
\]

or

\[
I(a_0, a_1, \omega) = \frac{\pi}{\omega} \sum_{j=0}^{\infty} a_j^2 \left( j^2 \omega^2 - k \right)
\]

Want to find \( a_1, a_2, \ldots \) s.t. \( I \) is an extremum (Hamilton's principle)

\[
\frac{\partial I}{\partial a_l} = 0 \Rightarrow \frac{\pi}{\omega} \sum_{j=0}^{\infty} 2 a_j \delta_{j,l} \left( j^2 \omega^2 - k \right) \quad \text{for } l=0, 1, 2, \ldots
\]

or

\[ a_l \left( j^2 \omega^2 - k \right) = 0 \quad \text{for } l=0, 1, 2, \ldots \]

\( l=0 \) gives \( a_0 = 0 \).

\( l \neq 0 \) gives either \( j \omega = \sqrt{\frac{k}{m}} \) or \( a_l = 0 \)

Take \( \omega = \sqrt{\frac{k}{m}} \) then \( a_1 \neq 0 \) and \( a_l = 0 \) for \( l \neq 1 \).
Goldstein, Po, Safko, 3rd Ed. Ch. 2 Prob. 19 pg 67 — Solution.

A particle of mass \( m \) at position \( \mathbf{r} \) is interacting with mass that is fixed in space with various density distributions \( \rho(\mathbf{r'}) \). The interaction with \( d\mathbf{m}' = \rho(\mathbf{r'}) d^3r' \)

is a potential depending only on \( |\mathbf{r} - \mathbf{r}'| \).

Thus, the total potential energy is given by

\[
V(\mathbf{r}) = \int U(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r'}) d^3r'
\]

(a) Mass \( \rho(\mathbf{r'}) \) is uniformly distributed in the plane \( z = 0 \).

Then \( V \) only depends on the distance from the plane.

This implies, using cartesian coords,

\[
V(z) \propto \text{no } x \text{ or } y \text{ dependence.}
\]

\[L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \Rightarrow \text{translational sym. in } x \text{ and } y\]

\[L(\dot{x}, \dot{y}, \dot{z}, x, y, z) \Rightarrow \frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} = m \dot{x} = \text{constant} = P_x\]

\[L \text{ does not depend on } x, \text{ or } y. \Rightarrow \frac{\partial L}{\partial \dot{y}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{y}} = m \dot{y} = \text{constant} = P_y\]

Also, \( L(z') \), (\( L \) does not depend explicitly on \( t \))

\[
\text{thus } \quad h = \dot{x} \frac{\partial L}{\partial \dot{x}} + \dot{y} \frac{\partial L}{\partial \dot{y}} + \dot{z} \frac{\partial L}{\partial \dot{z}} - L = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V = \text{conserved.}
\]

\[h = T + V = E = \text{constant} \]

Jacobi's Total mechanical Integral \( h = \text{Energy is conserved.} \)
(b) Mass is distributed on the half-plane $z = 0, y > 0$.

Now $V(y, z) = L(x, t)$ (does not depend on $x$ or $t$.)

Thus, translational symmetry in $x \Rightarrow \frac{\partial}{\partial x} = 0 \Rightarrow \frac{\partial}{\partial x} = \text{const.}$

still have $L(x) \Rightarrow \hat{h} = E = \text{const.}$

(c) Mass uniformly distributed in circular cyl. of $\infty$ length.

Use cylindrical polar coordinates: $(r, \phi, z)$

\[ V(r) = T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + \frac{1}{2} m z^2 \]

\[ L = T - V \Rightarrow L(A, \vec{r}, \vec{\pi}) \text{ independent of } \phi, z, \phi, \pi. \]

\[ L(r) \Rightarrow \hat{h} = \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial \dot{r}} = T + V = E = \text{const.} \]

\[ L(\phi) \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \hat{p}_\phi \text{ independent of } \phi, z, \phi, \pi. \]

\[ \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \dot{\phi}} = m \dot{\phi} = \text{const.} = \hat{p}_\phi \]

\[ L(z) \Rightarrow \hat{p}_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} = \text{const.} = \hat{p}_z \]

(d) Same mass dist. as (c) but now cyl. is finite length.

This leads to $V(r, z)$ but does not depend on $\phi$.

\[ \Rightarrow \hat{h} = E = \text{const.} = E_0 \]

\[ \hat{p}_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const.} = \hat{p}_\phi \]

(e) Same as (c) but cross section is ellipse. $\Rightarrow V(r, \phi)$

\[ \Rightarrow \hat{h} = E = E_0 = \text{const.} \]

\[ \hat{p}_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} = \text{const.} = \hat{p}_z \]
(g) mass is distributed on an infinite helical solenoid wound around the \( z \) axis. — Use cyl. polar coords.
Assume eq. for the helical wire is \( \phi = k z \), \( r = b \).

Try to think of a transformation that leaves the physical situation unchanged, i.e., the same distance from mass on the helical wire solenoid.
Clearly, \( V \) will depend on \( r \). So must keep \( r \) fixed.

If start at position \( \phi \) and \( z \). Then if move to new position \( z' = z + \Delta z \) and \( \phi' = \phi + k \Delta z \) the solenoidal mass distribution will look exactly the same as before and hence the potential energy must be the same. Thus, the potential energy function must be of the form
\[ V = V(r, z) \text{ where } g = g(\phi - kr). \]

The \( \varepsilon \)-family \( \eta \)-transformation:
\[ r'(\varepsilon ; \eta) \equiv r'^{-} = r' \quad \phi' = \phi + k \varepsilon \]
\[ z'(\varepsilon ; \eta) \equiv z'^{-} = z' - \varepsilon \]

The \( \varepsilon \)-family is the transformation:
\[ r'(\varepsilon ; \eta) \equiv r'^{-} = r' \quad \phi' = \phi + k \varepsilon \]
\[ z'(\varepsilon ; \eta) \equiv z'^{-} = z' - \varepsilon \]
Since \( \phi(\phi'; z') = \phi' - kE - kZ' + kE = \phi' - kZ' \)
we see that \( V' = V(\phi', \phi', z'; \phi', z') = V(\phi'; \phi', z') \approx \text{indep. of } \epsilon \).

Thus \( L' = L(\phi', \phi', z', \phi', z') \)
\( = L(\phi', \phi', z', \phi', z') \) independent of \( \epsilon \).

\[ \frac{\partial L'}{\partial \epsilon} = 0 \Rightarrow \frac{\partial L'}{\partial \epsilon}_{\epsilon=0} = 0 \]

and \( L \) is invariant under this family of \( \gamma \)-transformations.

Using Noether's theorem, there is a corresponding constant of motion:

\[ K = \frac{\partial L}{\partial \epsilon} \frac{\partial \epsilon}{\partial \epsilon}^{10} + \frac{\partial L}{\partial \phi} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial z} \frac{\partial z}{\partial \epsilon} = \text{const.} \]

\[ K = 0 + \frac{\partial L}{\partial \phi} (-k) + \frac{\partial L}{\partial z} (-1) = \text{const.} \]

\[ K = -\frac{\partial L}{\partial \phi} k - \frac{\partial L}{\partial z} = \text{const.} \]

\[ -K = k P_\phi + \mu P_z = \text{const.} \]

or

\[ k m r^2 \phi + \mu z = \text{const.} \]