Problem 8-1: [10 pts] Consider a tetrahedral molecule consisting of four identical atoms of mass $m$ positioned at the vertices of a tetrahedron of side $A$. Assume the molecule is held together by identical harmonic forces for small oscillations and that the force constants are all given by $k$. Thus, we have four equal masses at the vertices of a tetrahedron and identical springs with unstretched length $A$ along the edges connected to a mass at each end. Three springs connect to each mass and there are six springs in all. Each atom of the molecule can move in three dimensions.

(a) Without writing any equations of motion, give the total number of normal modes for this molecule and how many of these modes have zero frequency. Give clear brief explanations of your reasoning that justifies your answer in each case.

(b) Finding all the remaining normal modes is too time consuming for a homework problem, but one of the remaining normal modes corresponds to a symmetrical stretching of along the centerline of the tetrahedron at each of the four vertices of the molecule (a sort of breathing mode). Write a sentence explaining why we would expect this motion to be a normal mode. Find the frequency of this mode. Note: The angle $a$ between each edge at a vertex and the centerline at that vertex is such that $\cos a = \sqrt{2}/3$. See diagram on the right.

Problem 8-2: [20 pts] Consider the coupled torsional pendulum - spring system demonstrated in class. A reasonable mathematical model for the system has potential energy of the form

$$V = \frac{1}{2} kx^2 + \frac{1}{2} \theta^2 + \epsilon x \theta,$$

where $x$ is the downward displacement of the mass from its equilibrium hanging position, $\theta$ is the angular rotation of the mass from its angle when hanging in its equilibrium position, and the coupling $\epsilon$ is assumed to be small.

(a) Give a brief physical justification for the coupling between $x$ and $\theta$ used in the model.

(b) Find the normal mode frequencies and the normal modes. Give a brief physical description of the normal modes.

(c) Consider initial conditions where all velocities are zero, the spring is slightly compressed from its equilibrium length, and the torsional pendulum is at its equilibrium position. Show that your model leads to the “beat” phenomena as demonstrated in class. Hint: Note that the system used in the class demonstration had the parameters adjusted such that the two normal mode frequencies were very nearly equal.

Problem 8-3: [20 pts] A light string of length $3a$ is suspended from one end and has particles of masses $6m$, $2m$, and $m$ attached to it at distances $a$, $2a$, $3a$ respectively from the top end. Consider small oscillations from equilibrium for this triple pendulum constrained to oscillate in a vertical plane through its point of suspension.

(a) Find the kinetic energy matrix $\mathbf{T}$ and the potential energy matrix $\mathbf{V}$ for this system.

(b) How many normal modes does this system have (include your reason for your answer)? Find the eigenfrequencies of this triple pendulum for small oscillations and clearly show how you obtained your answer.

Note: We can simplify the characteristic equation by manipulating the matrix of coefficients such that it is triangular. This can be done using the following properties of determinants:

Starting with matrix $A$,

(i) $\det A$ is unchanged by replacing any row (or column) with the sum of that row (or column) with any other row (or column).

(ii) Multiplying any row (or column) by a constant $c$ also multiplies the $\det A$ by $c$.

[Answer to (b): $w_1^2 = 3g/a$, $w_2^2 = 3g/(2a)$, and $w_3^2 = g/(2a)$].

(c) Find the un-normalized normal mode displacement vector for each normal mode, $a_i$, and give a clear, but brief, physical description of each normal mode.
Tetrahedral molecule problem — solution

\[ m_1 = m_2 = m_3 = m_4 = m. \]

Therefore, must have 12 normal modes.

**How many normal modes have \( \omega = 0 ? \)**

\( \omega = 0 \) if there is no restoring force.

There will be no restoring force for rigid body displacements of the molecule. (There are 6 in all.)

**Translations (of C.M.):**

\( \omega_1 = 0 \) \( \Leftrightarrow \) - along \( x \)-axis: \( x_1 = x_2 = x_3 = x_4 = x \), \( y_1 = y_2 = y_3 = y_4 = y \), \( z_1 = z_2 = z_3 = z_4 = z \).

\( \omega_2 = 0 \) \( \Leftrightarrow \) - along \( y \)-axis: \( y_1 = y_2 = y_3 = y_4 = y \), \( x_1 = x_2 = x_3 = x_4 = x \), \( z_1 = z_2 = z_3 = z_4 = z \).

\( \omega_3 = 0 \) \( \Leftrightarrow \) - along \( z \)-axis: \( z_1 = z_2 = z_3 = z_4 = z \), \( x_1 = x_2 = x_3 = x_4 = x \), \( y_1 = y_2 = y_3 = y_4 = y \).

**Rotations (about C.M.):**

\( \omega_4 = 0 \) \( \Leftrightarrow \) - rotation about \( x \)-axis

\( \omega_5 = 0 \) \( \Leftrightarrow \) - rotation about \( y \)-axis

\( \omega_6 = 0 \) \( \Leftrightarrow \) - rotation about \( z \)-axis

Therefore, there are 6 normal modes with \( \omega = 0 \) and 6 modes with \( \omega \neq 0 \).
Tetrahedral molecule problem — solution continued.

Consider the "breathing" mode:

\[ \Delta Z = \text{displacement of m along centerline of a vertex,} \]

This same displacement is made at each vertex. The result is that all edges of the tetrahedron change by the same amount \( \Delta l \). So the molecule simply oscillates in size.

This displacement is a normal mode because of symmetry, all springs stretch equally, and there will be no forces \( \perp \) to the centerline. So the restoring force will always be directed directly back along the centerline toward the equilibrium position. No new motion along any other directions would be induced \( \Rightarrow \) a normal mode.

Find the angular frequency, \( \omega \), of this "breathing" mode:

Method 1: From geometry of the tetrahedron. Given \( \cos \alpha = \frac{1}{3} \)

\[ \frac{\Delta l}{Z} = \Delta Z \cos \alpha \quad \Rightarrow \quad F = k \Delta l \]

\[ m \Delta Z = F_{\perp} = -3F \cos \alpha = -3k \Delta l \cos \alpha = -6k \Delta Z \cos \alpha \]

\[ \text{or} \quad m \Delta Z = -6k \cos ^2 \alpha \Delta Z \Rightarrow \text{SiH}_4 \text{ with } \omega ^2 = \frac{6k}{m} \]

Method 2: (Lagrangian method)

\[ F = 4 \times \frac{1}{2} m (\Delta Z)^2 \quad V = 6 \times \frac{1}{2} k (\Delta l)^2 = 3k \times 4 \cos ^2 \alpha (\Delta Z)^2 = 8k (\Delta Z)^2 \]

\[ L = T-V = \frac{1}{2} m (\Delta Z)^2 - 8k (\Delta Z)^2 \]

Gives Lagrange eq. for \( \Delta Z \):

\[ m \Delta Z = -4k \Delta Z \Rightarrow \text{SiH}_4 \text{ with } \omega ^2 = 4k \]
(a) Mathematical model for coupled linear & torsional spring-mass system
used in class demonstration.

\[ T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \]

\[ V = \frac{1}{2} k'(x-x_0)^2 + \frac{1}{2} \beta'(\theta - \theta_0)^2 \]

The length of the wire wound in a cylinder of radius \( R \) is fixed. Thus, when the spring is extended, the wire spring wound in the cylindrical shape will not wrap around quite as far as it did previously. Thus, the equilibrium \( x_0 \) will change: \( x_0 = -\alpha x \), where \( \alpha \) is constant.

For the same reason, if the spring is twisted by \( \theta \) the equilibrium position will change: \( x_0 = -c \theta \), where \( c = \text{const} \).

Subst. back into the expression above for \( V \) gives:

\[ V = \frac{1}{2} k'(x+c \theta)^2 + \frac{1}{2} \beta'(\theta + \alpha x)^2 \]

or

\[ V = \frac{1}{2} (k + \beta \alpha^2) x^2 + \frac{1}{2} (\beta' + k' c^2) \theta^2 + (k' c + \beta' \alpha) x \theta \]

or

\[ V = \frac{1}{2} k x^2 + \frac{1}{2} \beta \theta^2 + \epsilon x \theta \]

where \( k = k' + \beta \alpha^2 \), \( \beta = \beta' + k' c^2 \), \( \epsilon = k' c + \beta \alpha \).
Demonstrating coupled spring + torsion oscillations.

(b) Determine normal mode freq. and describe normal modes.

\[ \begin{align*}
T &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \\
V &= \frac{1}{2} k x^2 + \frac{1}{2} \beta \theta^2 + \varepsilon x \theta \\
\end{align*} \]

Assume \( \frac{k}{m} = \frac{\beta}{I} = \omega_0^2 \Rightarrow k = \omega_0^2 m \) and \( \beta = \omega_0^2 I \)

Find freq. & normal modes:

Also: Assume \( \frac{\varepsilon}{\sqrt{m I}} \ll \frac{\beta}{I} = \omega_0 \)

\[
\begin{align*}
\text{det}([T] - \varepsilon [I]) &= 0 \\
\begin{vmatrix}
\varepsilon - \omega_0^2 & \varepsilon \\
\varepsilon & -\varepsilon + \omega_0^2
\end{vmatrix}
\end{align*}
\]

\[
(k-m\varepsilon^2)(\beta+\varepsilon^2) = \varepsilon^2 = 0
\]

\[
m\varepsilon^2 (\omega_0^2 - \omega_0^2) = \varepsilon^2 = 0
\]

\[
(\omega_0^2 - \omega^2) = \varepsilon^2 = 0 = (\omega_0^2 - \omega^2)(\omega_0^2 + \frac{\varepsilon^2}{\sqrt{m I}} - \omega^2)
\]

\[
\omega^2 = \frac{\omega_0^2 + \varepsilon^2}{2}
\]

\[
\begin{align*}
\alpha_1 &= \begin{pmatrix} 1 \\ \frac{\varepsilon}{\sqrt{m I}} \end{pmatrix} \\
\alpha_2 &= \begin{pmatrix} -\frac{\varepsilon}{\sqrt{m I}} \\ 1 \end{pmatrix}
\end{align*}
\]

\[
\alpha_1 = \alpha_{x1} + \frac{\varepsilon}{\sqrt{m I}} \quad \alpha_2 = \alpha_{x2} - \frac{\varepsilon}{\sqrt{m I}}
\]

\[
\begin{align*}
\alpha_{x1} &= \begin{pmatrix} 1 \\ \frac{\varepsilon}{\sqrt{m I}} \end{pmatrix} \\
\alpha_{x2} &= \begin{pmatrix} -\frac{\varepsilon}{\sqrt{m I}} \\ 1 \end{pmatrix}
\end{align*}
\]

\[
\alpha_{xj} = \alpha_{xj} \sqrt{\frac{m I}{\varepsilon}}
\]
Most students did not assume $k = \frac{\varepsilon}{I}$ at the outset.

So, I include the following:

Let $\omega_k \equiv \frac{k}{m}$ and $\omega_\beta \equiv \frac{\beta}{I}$.

Then $[T] = mI \begin{bmatrix} \frac{1}{I} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$ and $[V] = mI \begin{bmatrix} \frac{\omega_k - \omega}{I} & \frac{\varepsilon}{mI} \\ \frac{\varepsilon}{mI} & \frac{\omega_\beta - \omega}{m} \end{bmatrix}$.

$|V - \omega^2 T| = 0$ gives

$\begin{vmatrix} \frac{\omega_k^2 - \omega^2}{I} & \frac{\varepsilon}{mI} \\ \frac{\varepsilon}{mI} & \frac{\omega_\beta^2 - \omega^2}{m} \end{vmatrix} = 0$

or

$\frac{1}{mI} (\omega_k^2 - \omega^2)(\omega_\beta^2 - \omega^2) - \frac{\varepsilon^2}{(mI)^2} = 0$

or

$(\omega_k^2 - \omega^2)(\omega_\beta^2 - \omega^2) - \frac{\varepsilon^2}{mI} = 0$

or

$\omega^4 - (\omega_k^2 + \omega_\beta^2)\omega^2 + \omega_k^2\omega_\beta^2 - \frac{\varepsilon^2}{mI} = 0$

Thus (after some algebra)

$\omega_1^2 = \frac{(\omega_k^2 + \omega_\beta^2) \pm \sqrt{[\omega_k^2 - \omega_\beta^2]^2 + \frac{\varepsilon^2}{mI}}}{2}$
Normal modes:

\[ \omega_1^2 = \left( \frac{\omega_p^2 + \omega_k^2}{2} \right) + \sqrt{\left( \frac{\omega_p^2 - \omega_k^2}{2} \right)^2 + \left( \frac{\epsilon^2}{m \gamma} \right)^2} \]

where \( \omega_p^2 = \frac{k}{m} \) and \( \omega_k^2 = \frac{E}{m \gamma} \)

\[ a_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \]

or

\[
O = \begin{pmatrix} k - \mu \omega_1^2 & \epsilon \\ \epsilon & \rho - i \omega_1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} (k - \mu \omega_1^2) a_{11} + \epsilon a_{21} = 0 \\ \rho - i \omega_1 a_{21} \end{pmatrix}
\]

\[ a_{21} = -\left( \frac{k - \mu \omega_1^2}{\rho - i \omega_1} \right) a_{11} \]

or

\[ a_{21} = -\left( \frac{k - \mu \omega_1^2}{\rho - i \omega_1} \right) a_{11} \]

Thus \( a_1 = a_{11} \begin{pmatrix} 1 \\ -\frac{(\omega_k^2 - \omega_1^2) m}{\epsilon} \end{pmatrix} \)

Similarly for mode 2:

\[ \omega_2^2 = \left( \frac{\omega_p^2 + \omega_k^2}{2} \right) + \sqrt{\left( \frac{\omega_p^2 - \omega_k^2}{2} \right)^2 + \epsilon^2} \]

\[ a_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \quad \text{(same as above but replace } a_{11} \text{ with } a_{22}) \]

\[ a_2 = a_{12} \begin{pmatrix} 1 \\ -\frac{(\omega_k^2 - \omega_2^2) m}{\epsilon} \end{pmatrix} \]

Note: If assume \( \omega_k^2 = \omega_p^2 = \omega_0^2 \) then \( \omega_1^2 = \omega_0^2 + \frac{\epsilon^2}{m \gamma} \)

and \( a_1 = a_{11} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{m \gamma}} \end{pmatrix} \)

and \( a_2 = a_{12} \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{m \gamma}} \end{pmatrix} \)

For part (c) should assume \( \omega_k^2 = \omega_p^2 = \omega_0^2 \) and my original solution stands.
Coupled

Demo: Spring + torsion bars:

(c) show boat phenomena

Normalization:

$\alpha_1^T [T] \alpha_1 = 1$

$\alpha_{x1} \frac{1}{\sqrt{m}} \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \alpha_{x1} \frac{1}{\sqrt{m}} = \alpha_{x1} \frac{1}{\sqrt{m}} \frac{m}{\sqrt{m}} = \alpha_{x1} 2m = 1$

$\alpha_{x1} = \frac{1}{2m} \Rightarrow \alpha_{x1} = \frac{1}{\sqrt{2}m}$

$\alpha_1 = \left( \begin{array}{c} \frac{1}{\sqrt{2}m} \\ \frac{1}{\sqrt{2}m} \end{array} \right)$

Similarly $\alpha_2^T [T] \alpha_2 = 1 \Rightarrow \alpha_{x2} = \frac{1}{\sqrt{2}m}$

$\alpha_2 = \left( \begin{array}{c} \frac{1}{\sqrt{2}m} \\ -\frac{1}{\sqrt{2}m} \end{array} \right)$

$e^{iwt} = \cos wt - i \sin wt$

Gen. Disp. $\{ \varphi \} = \text{Re} \left( \sum \alpha_k C_k e^{iwt} \right) = \sum \alpha_k [\text{Re} C_k \cos wt + i \text{Im} C_k \sin wt]$

I.C. $A(0) = 0, x = x_0, \theta = 0 \Rightarrow \varphi = \left( \begin{array}{c} x_0 \\ 0 \end{array} \right)$

$\dot{\varphi} = 0 \Rightarrow \dot{\alpha} = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$

$\text{Re} C_1 = \alpha_1^T [T] \varphi_0 = \left( \frac{1}{\sqrt{2}m} \frac{1}{\sqrt{2}m} \right) \left( \begin{array}{c} m \theta \left( x_0 \right) \\ 0 \end{array} \right) = \sqrt{m} x_0$

$\text{Im} C_1 = \omega_1 \alpha_1^T [T] \varphi_0 = 0$

$\text{Re} C_2 = \alpha_2^T [T] \varphi_0 = \left( \frac{1}{\sqrt{2}m} \frac{1}{\sqrt{2}m} \right) \left( \begin{array}{c} m \theta \left( x_0 \right) \\ 0 \end{array} \right) = \sqrt{m} x_0$

$\text{Im} C_2 = 0$
Thus,
\[ M(t) = \begin{bmatrix} \frac{1}{\sqrt{2m}} \cos \omega_0 t + \frac{1}{\sqrt{2m}} \cos \omega_1 t \\ \frac{1}{\sqrt{2m}} \sin \omega_0 t + \frac{1}{\sqrt{2m}} \sin \omega_1 t \end{bmatrix} \]

or
\[ \psi(t) = \frac{1}{2} \begin{bmatrix} x_0 \cos \omega_0 t + \cos \omega_1 t \\ \frac{1}{\sqrt{2}} x_0 \sin \omega_0 t + \sin \omega_1 t \end{bmatrix} \]

Using \[ \cos A + \cos B = \cos \frac{A+B}{2} \cos \frac{A-B}{2} \] and \[ \cos A - \cos B = -2 \sin \frac{A-B}{2} \sin \frac{A+B}{2} \],

we have
\[ \psi(t) = \begin{bmatrix} x_0 \cos \left( \frac{\omega_0 - \omega_1}{2} t \right) + \cos \left( \frac{\omega_0 + \omega_1}{2} t \right) \\ \frac{1}{\sqrt{2m_1}} \sin \left( \frac{\omega_0 - \omega_1}{2} t \right) \pm \sin \left( \frac{\omega_0 + \omega_1}{2} t \right) \end{bmatrix} \]

\[ \frac{\epsilon}{\sqrt{m_1}} \ll \omega_0 \]

\[ \omega_1 = \omega_0 + \frac{\epsilon}{\sqrt{m_1}} \Rightarrow \omega_1 = \omega_0 \left( 1 + \frac{\epsilon}{\omega_0 \sqrt{m_1}} \right) \approx \omega_0 + \frac{1}{2} \frac{\epsilon}{\omega_0 \sqrt{m_1}} \]

\[ \omega_2 = \omega_0 - \frac{\epsilon}{\sqrt{m_1}} \Rightarrow \omega_2 = \omega_0 \left( 1 - \frac{\epsilon}{\omega_0 \sqrt{m_1}} \right) = \omega_0 - \frac{1}{2} \frac{\epsilon}{\omega_0 \sqrt{m_1}} \]

\[ \frac{\omega_1 - \omega_2}{2} = \frac{\epsilon}{2 \omega_0 \sqrt{m_1}} \ll \omega_0 \text{ and } \frac{\omega_1 + \omega_2}{2} = \omega_0 \]

Then
\[ \varphi(t) = \begin{bmatrix} x_0 \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \cos \omega_2 t \\ \frac{1}{\sqrt{2}} x_0 \sin \left( \frac{\omega_1 - \omega_2}{2} t \right) \sin \omega_2 t \end{bmatrix} \]

\[ \frac{\epsilon}{\omega_2 \sqrt{m_1}} \ll \omega_0 \]

Show the beat phenomenon:

A motion is out of phase with the motion. Shows the transfer of KE from translational to rotational and back.
Triple pendulum problem: Solution

Three generalized coordinates: $\Theta_1, \Theta_2, \Theta_3$.

Equilibrium position: $\Theta_1 = \Theta_2 = \Theta_3 = 0$.

Small oscillations $\Rightarrow \Theta_1, \Theta_2, \Theta_3 \ll 1$

(a) Keep terms up to $O(\Theta^2)$ (including $\dot{\Theta}_1, \dot{\Theta}_2, \dot{\Theta}_3$)

\[ m_1 = 6m: \]
\[ x_1 = a \sin \Theta_1 = a \dot{\Theta}_1 + O(\Theta^3); \quad \dot{x}_1 = a \ddot{\Theta}_1 + O(\Theta^4) \]
\[ y_1 = -a \cos \Theta_1 = -a (1 - \frac{\Theta_1^2}{2}) + O(\Theta^4); \quad \dot{y}_1 = a \dot{\Theta}_1 \dot{\Theta}_1 + O(\Theta^4) \]

\[ m_2 = 2m: \text{ (to order } \Theta^2) \]
\[ x_2 = x_1 + a \Theta_2 = a (\dot{\Theta}_1 + \dot{\Theta}_2); \quad \dot{x}_2 = a (\ddot{\Theta}_1 + \ddot{\Theta}_2) \]
\[ y_2 = y_1 - a (1 - \frac{\Theta_2^2}{2}); \quad \dot{y}_2 = a (\dot{\Theta}_1 \dot{\Theta}_2 + \Theta_1 \ddot{\Theta}_2) \]

\[ m_3 = m: \]
\[ x_3 = x_2 + a \Theta_3 = a (\dot{\Theta}_1 + \dot{\Theta}_2 + \dot{\Theta}_3); \quad \dot{x}_3 = a (\ddot{\Theta}_1 + \ddot{\Theta}_2 + \ddot{\Theta}_3) \]
\[ y_3 = y_2 - a (1 - \frac{\Theta_3^2}{2}); \quad \dot{y}_3 = a (\dot{\Theta}_1 \dot{\Theta}_2 + \dot{\Theta}_1 \dot{\Theta}_3 + \Theta_1 \ddot{\Theta}_2 + \Theta_1 \ddot{\Theta}_3) \]

Potential Energy:

\[ V = g (m_1 y_1 + m_2 y_2 + m_3 y_3) = mg a \left( \frac{9 \Theta_1^2}{2} + 3 \frac{\Theta_2^2}{2} + \frac{\Theta_3^2}{2} \right) + \text{const.} \]

or

\[ V = \frac{1}{2} \Theta^T [V] \Theta \quad \text{where} \quad \Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix} \quad [V] = mga \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Kinetic Energy:

\[ T = \frac{1}{2} \left[ m_1 (x_1^2 + y_1^2) + m_2 (x_2^2 + y_2^2) + m_3 (x_3^2 + y_3^2) \right] \quad \text{but} \quad \dot{y}_3 = 0 + O(\Theta^4) \]

or

\[ T = \frac{m a^2}{2} \left( 3 \dot{\Theta}_1^2 + 2 \dot{\Theta}_2^2 + 4 \dot{\Theta}_1 \dot{\Theta}_2 + 2 \dot{\Theta}_2^2 + \dot{\Theta}_3^2 + 2 \dot{\Theta}_1 \dot{\Theta}_3 + 2 \dot{\Theta}_2 \dot{\Theta}_3 + \dot{\Theta}_3^2 \right) \]

or

\[ T = \frac{1}{2} \dot{\Theta}^T [T] \dot{\Theta} \quad \text{where} \quad [T] = m a^2 \begin{bmatrix} 9 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \dot{\Theta}_{\text{equl}} \]
Triple pendulum problem. Solution continued.

(b) How many normal modes are there? What are the N.M. freqs?

From small-osc. theory, the no. of normal modes is equal to the no. of degrees of freedom, which is equal to the number of generalized coordinates required to describe the system. We have $\theta_1, \theta_2, \theta_3, \ldots$ have 3 normal modes.

Find the normal mode frequencies:

From small-osc. theory, the normal mode frequencies are the solutions to the characteristic eq.:

$$\det ([V] - \omega^2 [I]) = 0$$

From part (a): $[V] = m g\alpha \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[T] = m \alpha^2 \begin{bmatrix} 9 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Let $\lambda \equiv \omega^2$ and $\lambda_0 \equiv \omega_0^2 = \frac{g}{\alpha}$.

Then our characteristic eq. is:

$$\det ([V] - \lambda [I]) = m \alpha^2 \begin{vmatrix} 9(\lambda_0 - \lambda) & -3\lambda & -\lambda \\ -3\lambda & 3(\lambda_0 - \lambda) & -\lambda \\ -\lambda & -\lambda & (\lambda_0 - \lambda) \end{vmatrix} = 0$$

We can simplify the characteristic eqn. by manipulating the determinant using the following properties of determinants:

Starting with matrix $A$,

1. $|A|$ is unchanged by replacing any row (or column) with the sum of that row (or column) with any other row (or column).
ii) If you form a new matrix $A'$ by multiplying any row or column by a scalar $b$ then the value of the det. is simply multiplied by $b$.

\[ \omega^2 \equiv \lambda \] are solutions of

\[
\begin{vmatrix}
9(\lambda_0 - 2\lambda) & -3\lambda & -1 \\
-3\lambda & 3(\lambda_0 - \lambda) & -\lambda \\
-\lambda & -\lambda & (\lambda_0 - 2\lambda)
\end{vmatrix} = 0
\Rightarrow
D =
\begin{vmatrix}
9\lambda_0 & -3\lambda_0 & 0 \\
-3\lambda & 3(\lambda_0 - \lambda) & -\lambda \\
-\lambda & -\lambda & (\lambda_0 - 2\lambda)
\end{vmatrix}
\]

- Replace row 2 with (row 2 - row 3)
- Then multiply row 1 by $\frac{1}{3}$

\[
= (3\lambda_0 - 2\lambda) \left[ \begin{vmatrix} 3\lambda_0 & -\lambda_0 \end{vmatrix} \right] + \lambda_0 \left[ -2\lambda (\lambda_0 - \lambda) - \lambda_0 \lambda \right] = 0
\]

\[
= (3\lambda_0 - 2\lambda) \left[ (3\lambda_0 - 2\lambda)(\lambda_0 - \lambda) - \lambda_0 \lambda \right] + \lambda_0 \left[ -3\lambda_0 + 2\lambda \right] = 0
\]

\[
= (3\lambda_0 - 2\lambda) \left[ (3\lambda_0 - 2\lambda)(\lambda_0 - \lambda) - 2\lambda^2 \right] = 0
\]

\[
=(3\lambda_0 - 2\lambda)(3\lambda_0 - 2\lambda)(\lambda_0 - \lambda) - 2\lambda^2 = (3\lambda_0 - 2\lambda)(3\lambda_0 - \lambda)(\lambda - 2\lambda) = 0
\]

Thus, finally we have (largest to smallest)

\[
\begin{align*}
\lambda_1 &= \omega_1^2 = 3\lambda_0 = 3g/a \\
\lambda_2 &= \omega_2^2 = 3\lambda_0/2 = 3g/2a \\
\lambda_3 &= \omega_3^2 = \lambda_0/2 = 9/2a
\end{align*}
\]
Triple pendulum problem. Solution Continued.

(un-normalized)

(c) Find each normal mode displacement vector and give brief physical description of each.

\[ \omega_1^2 = 3g/a : \mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \]

\[
\left( \begin{bmatrix} V \\ -\omega_1^2 [I] \end{bmatrix} \right) \mathbf{a}_1 = 0 = \begin{bmatrix} -3 \lambda_1 & 3(\lambda_2 - \lambda_1) & -3 \lambda_1 \\ -3 \lambda_1 & 3 \lambda_2 & -3 \lambda_1 \\ -3 \lambda_1 & -3 \lambda_1 & 2 \lambda_1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = 0
\]

\[
\begin{bmatrix} -18 & -9 & -3 \\ -9 & -6 & -3 \\ -3 & -3 & -2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = 0 = -6a_{11} - 3a_{12} - a_{13} = 0
\]

Thus, \( a_{12} = 3a_{11} \).

Subst. \( a_{12} = -3a_{11} \) into first eqn \( \Rightarrow a_{13} = +3a_{11} \).

Thus, \( a_1 = a_{11} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \).

1 m mass always stays exactly below 6 m mass.

\[ \omega_2^2 = \frac{3g}{2a} : \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \]

\[
-\omega_2^2 \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -3 \lambda_2 & 3(\lambda_3 - \lambda_2) & -3 \lambda_2 \\ -3 \lambda_2 & 3 \lambda_3 & -3 \lambda_2 \\ -3 \lambda_2 & -3 \lambda_2 & 2 \lambda_2 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = 0
\]

\[
\begin{bmatrix} -9 & 3 \lambda_3 & -9 \\ -3 \lambda_2 & 3 \lambda_3 & -9 \\ -3 \lambda_2 & -3 \lambda_2 & 2 \lambda_2 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = 0
\]

Thus, \( a_{22} = 0 \), \( a_{32} = 3a_{12} \).

Thus, \( a_2 = a_{12} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \).

2 m always stays exactly below 6 m.

Similarly for \( \omega_3^2 = \frac{g}{2a} \).

Find \( a_3 = a_{13} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).