**Problem 1:** [45 pts.] A cylinder of radius $R$, mass $M_1$, and moment of inertia (about its central axis) $I = \frac{1}{2}M_1 R^2$ is rolling without slipping on an incline with angle $\alpha$ with respect to the horizontal. The incline itself has mass $M_2$ and is free to slide without friction on a horizontal surface. Use $s$ and $x_2$ as generalized coordinates as shown in the figure below.

![Diagram](attachment:diagram.png)

(a) [10 pts] Clearly show that the Lagrangian is given by:

$$L = \frac{1}{2}(M_1 + M_2)x_2'^2 + \frac{3}{4}M_1 s^2 + M_1 (\cos \alpha)s x_2' - M_1 g (\sin \alpha)s$$

(b) [8 pts] Find any constants of motion and indicate the physical quantity each constant represents.

(c) [8 pts] Find the differential equations of motion for this system. (You do not need to solve these equations!)

(d) [7 pts] Now suppose a new constraint is added that forces $x_2$ to vary with time as: $x_2(t) = A \sin (\omega t)$, where $A$ and $\omega$ are given constants. Find the new Lagrangian.

(e) [7 pts] Using the new Lagrangian, find any constants of motion and the new equation(s) of motion.

(f) [5 pts] Go back to part (d) and use a Lagrange multiplier, $\lambda$, to add the new constraint so as to obtain the constraint force associated with the new constraint. (Just set up the equations, you do not need to solve them.)

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**Problem 2:** [25 pts.] A general surface of revolution may be described in cylindrical coordinates ($r, \phi, z$) by the function $r = r(z)$. The function $r(z)$ and its derivative $dr/dz \equiv r'(z)$ are given. We want to find the equation for the curve that is the shortest path between two points on this surface.

(a) [8 pts] Show clearly that the expression for the differential path length on the surface that is the result of displacement $dz$ and $d\phi$ is given by $(ds)^2 = (1 + r'^2)(dz)^2 + r'^2 (d\phi)^2$.

(b) [10 pts] Take $z$ as the independent variable and show that the curve, $\phi(z)$, that is the shortest path between two points on the surface is given by

$$\phi(z) = \phi_0 + k \int_{z_0}^{z} \frac{\sqrt{r'^2(z') + 1}}{r(z') \sqrt{r'^2(z') - k^2}} \, dz'.$$

(c) [7 pts] If the surface is a cylinder ($r = a$), show that the result given in part (b) is the equation for a helix.

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**Problem 3:** [30 pts.] This problem involves launching a satellite of mass $m$ into orbit about a spherical planet (with no atmosphere) of mass $M >> m$ and radius $\rho$. (Assume the mass of the planet is symmetrically distributed such that the potential $V(r) = -k/r$ applies for all $r > \rho$.) The mass $m$ is raised to height $h = R - \rho$ (i.e. to radius $R$) and given a velocity $\mathbf{v}$ perpendicular to the radius. Use the results from the two-body problem with Kepler potential: $r(\theta) = C/(1 + \epsilon \cos \theta)$, where $C = \ell^2/\mu k$ and $E = \frac{\mu k^2}{2} (\epsilon^2 - 1)$ to answer the following questions. Ignore the possibility of collision with the planet for the first two parts.

(a) [7 pts] Clearly show that the eccentricity $\epsilon$ as a function of $v \equiv |\mathbf{v}|$, is given by $\epsilon^2 = \left(\frac{Rv^2}{GM} - 1\right)^2$ where $G$ is the gravitational constant in Newton’s law of gravity.

(b) [8 pts] For what values of $v$ is the orbit an ellipse, a circle, a parabola, and a hyperbola?

(c) [7 pts] Find the smallest $v$ such that the satellite does not collide with the surface of the planet.

(d) [8 pts] Find the expression for $\epsilon$ if $\mathbf{v}$ is not perpendicular to the radius but at angle $\alpha$ to the perpendicular.