Chapter 7

Covariant Bilinear Forms

Through products of $\gamma$-matrices it is possible to form 16 linearly independent $4 \times 4$ matrices $\Gamma_{\alpha\beta}$ which often occur in Dirac theory. These matrices are

\[
\begin{align*}
\Gamma^s &= 1 \\
\Gamma^\nu_\mu &= \gamma_\mu \\
\Gamma^{\nu\mu} &= \sigma_{\mu\nu} \\
\Gamma^A_\mu &= \gamma_5 \gamma_\mu \\
\Gamma^p &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5 \equiv \gamma^5 
\end{align*}
\]  

(7.1)

Using the commutation relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1$ one can show that the $\Gamma^n$ are linearly independent. Brief arguments are

1) For each $\Gamma^n$ holds $(\Gamma^n)^2 = \pm 1$.

2) For each $\Gamma^n$ (except $\Gamma^s$) there is a $\Gamma^m$ such that

\[
\Gamma^n \Gamma^m = -\Gamma^m \Gamma^n .
\]  

(7.2)

From this follows that $tr \ \Gamma^n = 0$:

\[
\pm tr \ \Gamma^n = tr \ \Gamma^n (\Gamma^n)^2 = -tr \ \Gamma^m \Gamma^n \Gamma^m = -tr \ \Gamma^n (\Gamma^m)^2 = 0 .
\]  

(7.3)

3) For fixed $\Gamma^a$ and $\Gamma^b (a \neq b)$ there exists a $\Gamma^n \neq \Gamma^s$ such that

\[
\begin{align*}
\Gamma^a &\neq \Gamma^b \\
\Gamma^a &\neq \Gamma^s
\end{align*}
\]
\[ \Gamma^a \Gamma^b = \Gamma^n. \]  \hspace{1cm} (7.4)

4) Suppose there exist numbers \( a_n \) such that

\[ \sum_n a_n \Gamma^n = 0. \]  \hspace{1cm} (7.5)

Then via multiplication with \( \Gamma^n \neq \Gamma^s \) and taking the trace one finds because of (7.3) that \( a_m = 0 \). In the case of \( \Gamma^n = \Gamma^s \) one finds \( a_s = 0 \) and all coefficients vanish. This proves the linear independence of the \( \Gamma^n \). From this follows that every \( 4 \times 4 \) matrix can be expressed as a linear combination of the \( \Gamma^n \).