Chapter 8

The Dirac Current

As the next step, we need to show that the current density defined in (5.3) corresponds to the Dirac equation. Thus we have to show that \( \rho(x) \) constitutes the time-like component of the four vector \( j^\mu(x) \), whose divergence vanishes according to (4.7). We start from

\[
j^0(x) = \rho(x) = \sum_{\alpha=1}^{4} \psi^*_\alpha(x) \psi_\alpha(x) = \psi^\dagger(x) \psi(x)
\]

(8.1)

with

\[
\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) .
\]

(8.2)

To express clearly that \( \psi^\dagger \psi \) is a time-like component, we write

\[
j^0 = \psi^\dagger \psi = \psi^\dagger \gamma^0 \gamma^0 \psi = \bar{\psi} \gamma^0 \psi
\]

(8.3)

with

\[
\bar{\psi} := \psi^\dagger \gamma^0 ,
\]

(8.4)

the adjoint Dirac Spinor. Formally (8.3) can be generalized to a four-component quantity

\[
j^\mu := \bar{\psi} \gamma^\mu \psi .
\]

(8.5)

In the following we want to show that the above defined Dirac current is indeed divergence free, as well as that it transforms as four-vector under Lorentz transformations.
To calculate the divergence of (8.5), we need to consider

$$\partial_\mu j^\mu = \bar{\psi} \gamma^\mu (\partial_\mu \psi) + (\partial_\mu \bar{\psi}) \gamma^\mu \psi.$$  (8.6)

For the first term on the right-hand side, we use the Dirac equation directly:

$$(\gamma^\mu (i \partial_\mu - eA_\mu) - m) \psi = 0$$  (8.7)

gives

$$i \gamma^\mu \partial_\mu \psi = (\gamma^\mu eA_\mu + m) \psi.$$  (8.8)

For the second term, we start from the adjoint equation

$$\psi^\dagger (\gamma^\mu (-i \frac{\gamma^\mu}{\partial_\mu} + eA_\mu) - m) = 0.$$  (8.9)

Multiplication from the right with $\gamma^0$ and considering

$$\bar{\psi} \gamma^\mu := \gamma^0 \gamma^\mu \gamma^0$$  (8.10)

gives

$$\bar{\psi} (\bar{\gamma}^\mu (i \frac{\gamma^\mu}{\partial_\mu} + eA_\mu) - m) = 0.$$  (8.11)

Using the properties of the $\gamma$-matrices (6.22), (6.1, 6.2, 6.3) gives $\bar{\gamma}^\mu = \gamma^\mu$ and thus

$$\bar{\psi} (\gamma^\mu (i \frac{\gamma^\mu}{\partial_\mu} + eA_\mu) + m) = 0$$

$$i \partial_\mu \bar{\psi} \gamma^\mu = -\bar{\psi} (\gamma^\mu eA_\mu + m).$$  (8.12)

With (8.8) and (8.12) follows

$$i \partial_\mu j^\mu = \bar{\psi} (\gamma^\mu eA_\mu + m) \psi - \bar{\psi} (\gamma^\mu eA_\mu + m) \psi$$

$$= 0$$  (8.13)

Thus

$$\partial_\mu j^\mu = \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$  (8.14)
and the Dirac spinor can be used via the Dirac current as quantum mechanical probability amplitude.

We need to consider the space-like part of the probability current \( j^\mu \):

\[
\tilde{j} = (j^k) = \bar{\psi} \gamma_k \psi = \psi^\dagger \gamma^0 \gamma^k \psi = \psi^\dagger \tilde{\alpha} \psi
\]  

(8.15)

with

\[
\tilde{\gamma} := \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}
\]  

(8.16)

\[
\tilde{\alpha} := \gamma^0 \tilde{\gamma}.
\]  

(8.17)

From the explicit forms of \( \gamma^0 \) and \( \tilde{\gamma} \) follows

\[
\tilde{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \sigma & 0 \end{pmatrix},
\]  

(8.18)

thus \( \tilde{\alpha} \) consists of hermitian matrices. In addition, the trace vanishes, so that (8.15) only contains mixed products \( \psi_\alpha^* \psi_\beta \) with \( \alpha \neq \beta \).

**Gordon-Decomposition of the Dirac Current**

For \( A_\mu = 0 \) (8.8) and (8.12) gives

\[
\psi = \frac{1}{m} i \gamma^\nu \partial_\nu \psi
\]

\[
\tilde{\psi} = - \frac{1}{m} i \partial_\nu \tilde{\psi} \gamma^\nu
\]  

(8.19)

Inserting this into

\[
j^\mu = \frac{1}{2} \left( \bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu \psi \right)
\]  

(8.20)

gives

\[
j^\mu = \frac{1}{2m} \left( \bar{\psi} \gamma^\mu \gamma^\nu \left( \partial_\nu \psi \right) - \left( \partial_\nu \bar{\psi} \right) \gamma^\nu \gamma^\mu \psi \right).
\]  

(8.21)
Rewriting the products of $\gamma$-matrices as
\[
\begin{align*}
\gamma^\mu \gamma^\nu &= \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \\
\gamma^\nu \gamma^\mu &= \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} - \frac{1}{2} [\gamma^\mu, \gamma^\nu] 
\end{align*}
\] (8.22)

With the anti-commutation relations (5.20) and the definition of the spin tensor
\[
\sigma^{\mu \nu} := \frac{i}{2} [\gamma^\mu, \gamma^\nu] 
\] (8.23)

follows
\[
\begin{align*}
\gamma^\mu \gamma^\nu &= g^{\mu \nu} + \frac{1}{i} \sigma^{\mu \nu} \\
\gamma^\nu \gamma^\mu &= g^{\mu \nu} - \frac{1}{i} \sigma^{\mu \nu} .
\end{align*}
\] (8.24)

Inserting into (8.21) leads to
\[
j^\mu = \frac{i}{2m} \bar{\psi} \gamma^\mu \nabla^\mu \psi + \frac{1}{2m} \partial_\nu (\bar{\psi} \sigma^{\mu \nu} \psi) .
\] (8.25)

The first term is called "orbital current"
\[
j_{\text{orbit}}^\mu := \frac{i}{2m} \bar{\psi} \gamma^\mu \nabla^\mu \psi 
\] (8.26)

the second, the spin part of the current,
\[
j_{\text{spin}}^\mu := \frac{1}{2m} \partial_\nu (\bar{\psi} \sigma^{\mu \nu} \psi) .
\] (8.27)

Let us consider the spatial part $j^k$. With $\partial^k = -\partial_k = -(\nabla)_k$, the orbital part is given by
\[
\begin{align*}
j_{\text{orbit}}^k &= \frac{1}{2m} \bar{\psi} \gamma^0 (\nabla)_k \psi \\
&= \frac{1}{2m} \bar{\psi} \gamma^0 (\nabla)_k \psi
\end{align*}
\] (8.28)
This is obviously a generalization of the non-relativistic current (4.9), which we encountered with the Klein-Gordon equation, and will describe the probability current due to the spatial movement of an electron. For the second term, (8.27), we write

\[ j_{\text{spin}}^k = \frac{1}{2m} \partial_t (\bar{\psi} \sigma^{k\ell} \psi) + \frac{1}{2m} \partial_0 (\bar{\psi} \sigma^{k0} \psi) . \]  

(8.29)

The matrices \( \sigma^{k\ell} \) and \( \sigma^{k0} \) fulfill

\[ \sigma^{k\ell} = \epsilon_{k\ell m} \left( \begin{array}{cc} \sigma_m & 0 \\ 0 & \sigma_m \end{array} \right) := \epsilon_{k\ell m} \Sigma^m \]

\[ \sigma^{k0} = (-i) \left( \begin{array}{cc} 0 & \sigma_k \\ \sigma_k & 0 \end{array} \right) = (-i) \alpha^k \]

(8.30)

We introduced here the spin matrices \( \Sigma^m \), which describe the spin in a Dirac theory. Thus the first term on the right-hand side of (8.29) contains directly the spin. The second term is of relativistic nature, since here spatial and time components are mixed.