Phys. 611: Homework I

due January 16, 2004

1. Average values and Root-Mean-Square Deviation (2 pts each)
(a) The function of displacement of the classical harmonic oscillator is given by

\[ x(t) = A \sin(\omega t + \phi). \]

Calculate the average values

\[ \overline{x^2} = \frac{\int_0^T dt x^2(t)dt}{\int_0^T dt} \quad \text{and} \quad \overline{x} = \frac{\int_0^T dt x(t)}{\int_0^T dt} \]  

(1)

and the corresponding values for \( \overline{p^2} \) and \( \overline{p} \).

(b) Calculate the root-mean square deviation of \( x \) and \( p \), i.e.

\[ \Delta x = \sqrt{\overline{x^2} - \overline{x}^2} \quad \text{and} \quad \Delta p = \sqrt{\overline{p^2} - \overline{p}^2} \]  

(2)

(c) Using Bohr's quantum condition

\[ \int_0^T dx \ p = \int_0^T dt \ p \dot{x} = nh \]  

(3)

find the relation for \( \Delta x \cdot \Delta p \) and discuss the meaning of this result as far as classical mechanics and quantum mechanics are concerned.

2. Matrix Mechanics (2 pts each)
The Hamilton operator \( H \) of an harmonic oscillator is given by

\[ H = \frac{1}{2} P^2 + \frac{1}{2} X^2. \]  

(4)

In the Heisenberg formulation of quantum mechanics the operators \( P, X, \) and \( H \) are expressed as matrices. \( X \) and \( P \) satisfy the commutation relations

\[ [X, P] = i \mathbf{1} \]  

(5)

and define the matrix \( A = \frac{i}{\sqrt{2}}(P - iX) \) and its hermitian conjugate \( A^\dagger = \frac{1}{\sqrt{2}}(P + iX) \). Show that:

(a) the matrices \( A \) and \( A^\dagger \) satisfy the relation \( [A, A^\dagger] = \mathbf{1} \).
(b) the matrix $H$ can be expressed as

$$H = \frac{1}{2} (AA^\dagger + A^\dagger A).$$

(c) $[A, H] = A$ and $[A^\dagger, H] = -A^\dagger$.

d) starting from the relation

$$[X, P] = c1 \quad (c \in \mathbb{C})$$

the dimension of the matrices $X$ and $P$ has to be infinite.

e) the infinite dimensional matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$P = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

satisfy the relation $[X, P] = i1$.

(f) Find the eigenvalues of the matrix $H = \frac{1}{2} (P^2 + X^2)$ using the above definitions of $X$ and $P$, and compare them with the result of the usual solutions of the quantum mechanical harmonic oscillator functions.

3. Matrix Mechanics II (2 pts each)

Let $A$ be the infinite dimensional matrix

$$A_{nm} = \begin{cases} \sqrt{n} & \text{for } n + 1 = m \\ 0 & \text{otherwise} \end{cases} \quad n, m = 1, 2, 3, \ldots \infty,$$

and $N = A^\dagger A$.

(a) Show: (i) $[A, A^\dagger] = 1$, and (ii) $[A, N] = A$

(b) Calculate the elements of $N$ explicitly

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