1. Average values and Root-Mean-Square Deviation \((2 \text{ pts each})\)

(a) The function of displacement of the \textit{classical} harmonic oscillator is given by

\[ x(t) = A \sin(\omega t + \varphi). \]

Calculate the average values

\[ \bar{x}^2 = \frac{\int_0^T dt x^2(t) dt}{\int_0^T dt} \quad \text{and} \quad \bar{x} = \frac{\int_0^T dt x(t)}{\int_0^T dt} \quad (1) \]

and the corresponding values for \(\bar{p}^2\) and \(\bar{p}\).

(b) Calculate the root-mean square deviation of \(x\) and \(p\), i.e.

\[ \Delta x = \sqrt{\bar{x}^2 - \bar{x}} \quad \text{and} \quad \Delta p = \sqrt{\bar{p}^2 - \bar{p}} \quad (2) \]

(c) Using Bohr’s quantum condition

\[ \int_0^T dx p = \int_0^T dt p\dot{x} = nh \quad (3) \]

find the relation for \(\Delta x \cdot \Delta p\) and discuss the meaning of this result as far as classical mechanics and quantum mechanics are concerned.

2. Matrix Mechanics \((2 \text{ pts each})\)

The Hamilton operator \(H\) of an harmonic oscillator is given by

\[ H = \frac{1}{2} P^2 + \frac{1}{2} X^2. \quad (4) \]

In the Heisenberg formulation of quantum mechanics the operators \(P\), \(X\), and \(H\) are expressed as matrices. \(X\) and \(P\) satisfy the commutation relations

\[ [X, P] = i1 \quad (5) \]

and define the matrix \(A = \frac{i}{\sqrt{2}}(P - iX)\) and its hermitian conjugate \(A^\dagger = \frac{1}{i\sqrt{2}}(P + iX)\).

Show that:

(a) the matrices \(A\) and \(A^\dagger\) satisfy the relation \([A, A^\dagger] = 1\).
(b) the matrix $H$ can be expressed as

$$H = \frac{1}{2} \left( AA^\dagger + A^\dagger A \right).$$

(6)

(c) $[A, H] = A$ and $[A^\dagger, H] = -A^\dagger$.

(d) starting from the relation

$$[X, P] = c \mathbf{1} \quad (c \in \mathbb{C})$$

(7)

the dimension of the matrices $X$ and $P$ has to be infinite.

(e) the infinite dimensional matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\
0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\
0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\
0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

(8)

and

$$P = \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & \cdots \\
1 & 0 & -\sqrt{2} & 0 & 0 & \cdots \\
0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\
0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\
0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

(9)

satisfy the relation $[X, P] = i \mathbf{1}$.

(f) Find the eigenvalues of the matrix $H = \frac{1}{2}(P^2 + X^2)$ using the above definitions of $X$ and $P$, and compare them with the result of the usual solutions of the quantum mechanical harmonic oscillator functions.

3. Matrix Mechanics II (2 pts each)

Let $A$ be the infinite dimensional matrix

$$A_{nm} = \begin{cases} 
\sqrt{n} & \text{for } n+1 = m \\
0 & \text{otherwise}
\end{cases} \quad n, m = 1, 2, 3, \ldots \infty,$$

(10)

and $N = A^\dagger A$.

(a) Show: (i) $[A, A^\dagger] = \mathbf{1}$, and (ii) $[A, N] = A$

(b) Calculate the elements of $N$ explicitly