Phys. 611: Homework II

due January 19, 2005

1. Commutation Relations  (2 pts each)
Consider the relations between commutators of operators and their exponentials. Prove
that if $[P, Q] = \lambda 1$, then

(a) $e^{P+Q} = e^P e^Q e^{-\frac{\lambda}{2}}$

($Hint$: Consider operators $E(R) = e^{aR}$ and show that both sides of (a) satisfy the
same first order differential equation with respect to $a$.)

(b) $e^P e^Q = e^Q e^P e^{\lambda 1}$.

2. Normalized Eigenfunctions  (3 pts total)
Let $\phi_1$ and $\phi_2$ be the normalized eigenfunctions corresponding to the same eigenvalue. If

$$\int \phi_1^* \phi_2 d\tau = d,$$

where $d$ is real, find normalized linear combinations of $\phi_1$ and $\phi_2$ that are orthogonal to

(a) $\phi_1$

(b) $\phi_1 + \phi_2$.

3. Parity  (2 pts each)
Modern physics emphasizes the property of parity, i.e. whether a quantity remains invariant
or changes sign under an inversion of the coordinate system. In Cartesian coordinates
this means $(x, y, z) \rightarrow (-x, -y, -z)$.

(a) Show algebraically and geometrically that the inversion (reflection through the origin)
of a point $(r, \theta, \varphi)$ relative to fixed $x-$, $y-$, $z-$axes consists of the transformation

$$\begin{align*}
    r &\rightarrow r \\
    \theta &\rightarrow \pi - \theta \\
    \varphi &\rightarrow \varphi \pm \pi
\end{align*}$$

(1)
(b) Show that the unit vectors $\hat{r}$ and $\hat{\phi}$ have odd parity (reversal of direction) and that $\hat{\theta}$ has even parity.

4. **Parity Eigenfunctions** *(2 pts each)* Consider an real valued function $f(x)$ expanded in terms of even and odd functions

$$f(x) = c_E \tilde{f}_E(x) + c_O \tilde{f}_O(x),$$

where $\tilde{f}_E, O$ are normalized.

(a) Show that the probability that the wave function is in an even state is given by

$$c_E^2 = \frac{1}{2} \left( 1 + \frac{\int_{-\infty}^{\infty} dx \ f(x)f(-x)}{\int_{-\infty}^{\infty} dx \ f^2(x)} \right)$$

and in an odd state

$$c_O^2 = \frac{1}{2} \left( 1 - \frac{\int_{-\infty}^{\infty} dx \ f(x)f(-x)}{\int_{-\infty}^{\infty} dx \ f^2(x)} \right)$$

(b) Show explicitly that $c_E^2 = 1$ and $c_O^2 = 0$ for an even function.

5. **Root-Mean-Square Deviation** *(3 pts)*

Proof that in an eigenstate of $L_2$ and $L_3$ the root-mean-square deviation of the other two components is non-zero by calculating

$$(\Delta L_1)_{l,m} \quad \text{and} \quad (\Delta L_2)_{l,m}$$

*Hint:*

$$L_1 = \frac{1}{2}(L_+ + L_-), \quad L_2 = \frac{1}{i2}(L_+ - L_-)$$