Phys. 611: Homework IX

due March 9, 2005

1. Spin-1 particles

One description of spin-1 particles uses the matrices

\[
M_x = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0 \\
\end{pmatrix}
\quad M_y = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
i & 0 & 0 \\
\end{pmatrix}
\quad M_z = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

(1)

Hint: The elements of the matrices \( M \) can be expressed as \((M_k)_{lm} = -i\varepsilon_{klm}\) for \( k=1,2,3 \).

(a) (2 p) If the matrices \( M_i \) are the generators of a Lie algebra, determine the structure constants \( c^k_{ij} \) for which

\[
[M_i, M_j] = \sum_k c^k_{ij} M_k
\]

(2)

(b) (2 p) Find a Casimir operator \( c \) for which \([c, M_k] = 0\), and give \( c \) explicitly.

(c) (4 p) Show that the matrix-vector equation

\[
\left( \vec{M} \cdot \nabla + \frac{1}{c} \frac{\partial}{\partial t} \right) \Psi = 0
\]

(3)

reproduces Maxwell’s equations in the vacuum. Here \( \Psi \) is a column vector with components \( \Psi_j = B_j - iE_j/c, j = x, y, z \). \( \vec{M} \) is a vector whose elements are the angular matrices from above. Note that \( \varepsilon \mu_0 = 1/c^2 \).

5. Clebsch-Gordan Coefficients

(a) (3 p) Prove the following recursion relation for the Clebsch-Gordan coefficients:

\[
\sqrt{J(J+1) - M(M+1)} \langle j_1 m_1; j_2 m_2 | JM \pm 1 \rangle = \\
\frac{\sqrt{j_1 (j_1 + 1) - m_1 (m_1 - 1)} \langle j_1 m_1 ; j_2 m_2 | JM \rangle}{\sqrt{j_1 (j_1 + 1) - m_1 (m_1 + 1)} \langle j_1 m_1 + 1; j_2 m_2 | JM \rangle} + \\
\frac{\sqrt{j_2 (j_2 + 1) - m_2 (m_2 - 1)} \langle j_1 m_1 ; j_2 m_2 | JM \rangle}{\sqrt{j_2 (j_2 + 1) - m_2 (m_2 + 1)} \langle j_1 m_1 + 1; j_2 m_2 + 1 | JM \rangle}
\]

(Hint: use \( \langle j_1 m_1 ; j_2 m_2 | J_\pm | JM \rangle \).)
(b) \( (3p) \) Let \( j_1 = \frac{1}{2}, \) and \( j_2 = l. \) Use (a) and compute
\[
\langle \frac{1}{2} \pm \frac{1}{2}; l, m \mp \frac{1}{2}|l + \frac{1}{2}m \rangle = \sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}}
\] (5) \[\langle \frac{1}{2} \pm \frac{1}{2}; l, m \mp \frac{1}{2}|l - \frac{1}{2}m \rangle = \mp \sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}}
\] (6)

5. Pion-Nucleon Scattering

(a) \( (4p) \) Compute the Clebsch-Gordon coefficients for the states with \( J = J_1 + J_2, \)
\( M = m_1 + m_2, \) where \( j_1 = 1 \) and \( j_2 = 1/2, \) and \( j = 3/2, M = 1/2 \) for the various possible \( m_1 \) and \( m_2 \) values.

(b) \( (4p) \) Consider the reactions
\[
\begin{align*}
\pi^+ p & \rightarrow \pi^+ p \quad (7) \\
\pi^- p & \rightarrow \pi^- p \quad (8) \\
\pi^- p & \rightarrow \pi^0 n \quad (9)
\end{align*}
\]
These reactions, which conserve isospin, can occur in the isospin \( I = 3/2 \) state (\( \Delta \)-resonance) or the \( I = 1/2 \) state (\( N^* \) resonance). Calculate the ratios of these cross sections, \( \sigma(7) : \sigma(8) : \sigma(9), \) for an energy corresponding to the \( \Delta \)-resonance and the \( N^* \) resonance. At a resonance energy you can neglect the effect due to the other isospin states. Note that the pion is an isospin \( I = 1 \) state and the nucleon an isospin \( I = 1/2 \) state.