H2.1 Rotation Matrices and Parity Operators (3P)

a) A matrix $\mathbf{R}$ is a rotation if $\mathbf{R}^t \mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$. Show that a rotation in an odd dimensional space always leaves at least one non-zero factor fixed, i.e. $\mathbf{R}$ has 1 as eigenvalue. Give a geometrical interpretation of this result. (Hint: Consider $\det(\mathbf{R} - \mathbf{I})$)

b) Give an explicit representation of the parity operator $\mathbf{P}$ which maps the coordinates $(x, y, z) \rightarrow (-x, -y, -z)$ and discuss its relation to rotations in odd (even) dimensional spaces.

H2.2 Commutation relations (2P)

Consider relations between commutators of operators and their exponentials. Prove that if $[\mathbf{P}, \mathbf{Q}] = \lambda \mathbf{I}$ then

\[ e^{\mathbf{P} + \mathbf{Q}} = e^{\mathbf{P}} e^{\mathbf{Q}} e^{\frac{\lambda}{2}} \]

(Hint: Consider operators $E(\mathbf{R}) = e^{\mathbf{R}}$ and show that both sides of a) satisfy the same first order differential equation with respect to $\alpha$.)

b) \[ e^{\mathbf{P}} e^{\mathbf{Q}} = e^{\mathbf{Q}} e^{\mathbf{P}} e^{\frac{\lambda}{2}} \]

H2.3 Parity Operator (3P)

a) Explore parity and wave functions by using the Mathematica notebook PPSI. The notebook is programmed with the function

\[ \Psi(x) = \cos x + \left(\frac{x}{2} + 1\right) \sin x. \]

Run the program as is and discuss its output:

\[ \Psi_{\pm}(x) = \Psi(x) \pm \Psi(-x) \]
\[ \mathbf{P} \Psi_{\pm}(x) = \Psi_{\pm}(x) = 2 \cos x + x \sin x \]
\[ \mathbf{P} \Psi_{-}(x) = -\Psi_{-}(x) = -2 \sin x \]

Modify the definition of $\Psi(x)$ to a function of interest to you, but not necessarily of definite parity, and run the program to reproduce graphics for you to interpret.

H2.4 Root-Mean-Square Deviation (2P)

Proof that in an eigenstate of $\mathbf{L}^2$ and $\mathbf{L}_3$ the root-mean-square deviation of the other two components is non-zero, by calculating $(\Delta \mathbf{L}_1)_{l,m}$ and $(\Delta \mathbf{L}_2)_{l,m}$.

(Hint: $\mathbf{L}_1 = \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-)$, $\mathbf{L}_2 = \frac{1}{2}(\mathbf{L}_+ - \mathbf{L}_-)$)