Phys. 735: Homework I

due September 22, 2004

1. An operator $U(t + \varepsilon, t)$ describes the change in a wave function from $t$ to $t + \varepsilon$. For $\varepsilon$ real and small enough so that terms $\sim \varepsilon^2$ may be neglected

$$ U(t + \varepsilon, t) = 1 - \frac{i}{\hbar} \varepsilon H(t). $$

(a) (3 pts)
Show that $U$ is unitary, if and only if $H$ is hermitian.

(b) (2 pts)
Show that an alternate form

$$ U(t + \varepsilon, t) = \frac{1 - \frac{i}{\hbar} \varepsilon H(t)}{1 + \frac{i}{\hbar} \varepsilon H(t)} $$

agrees with the $U$ given above (provided terms of order $\varepsilon^2$ can be neglected) and is unitary if $H$ is hermitian.

2. Consider a particle of mass $m$ in the complex potential field

$$ \Phi = V(r) + \frac{i\hbar}{2} \omega(r), $$

where $V(r)$ and $\omega(r)$ are real functions.

(a) (4 pts)
What form does the continuity equation assume for this potential?

(b) (2 pts)
Offer an interpretation for the field $\omega(r)$.

(c) (2 pts)
Is the new continuity equation found in (a) time-reversible? (Prove your answer).

(d) (2 pts)
Is the related Hamiltonian Hermitian? (Prove your answer).
3. (10 pts)
For the simple harmonic oscillator the wave function of an energy eigenstate is given by

\[ u_n(x) \exp \left( -\frac{i}{\hbar} E_n t \right) = c_n \exp \left( -\frac{1}{2} \alpha^2 x^2 \right) H_n(\alpha x), \]

where

\[ \alpha = \sqrt{\frac{m\omega}{\hbar}} ; \quad c_n^2 = \frac{1}{2^n n! \sqrt{\pi}} ; \quad E_n = \hbar \omega \left( n + \frac{1}{2} \right) \]

Determine the propagator \( \mathcal{K}(x'', t; x', t_0) \), where you may consider \( t_0 = 0 \) for simplicity. Discuss the time behavior of the wave function.

Potentially useful formulae:

\[ \exp \left( -\lambda^2 + 2\lambda \eta \right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(\eta) \]

\[ \left( \frac{1}{\sqrt{1 - \gamma^2}} \right) \exp \left( -\frac{(\alpha^2 + \beta^2 - 2\alpha\beta\gamma)}{(1 - \gamma^2)} \right) = \exp[-(\alpha^2 + \beta^2)] \sum_{n=0}^{\infty} \left( \frac{\gamma^n}{2^n n!} \right) H_n(\alpha)H_n(\beta) \]