Phys. 735: Exam II (Take-Home)

due November 22, 2004, 11:00 am

I. Consider a simple model for hadronic structure, where the hadrons (protons, neutrons, mesons) occupy a spherical volume of radius R. Hadrons are assumed to be composed of elementary spin-1/2 particles called ‘quarks’. If a quark is inside this volume representing the hadron, its mass is assumed to be small, and it may be taken to zero. If it gets outside, interactions with the neighboring quarks, which make up the rest of the hadron, are assumed to generate an infinite mass for the quark.

To describe this model quantitatively, one must solve the Dirac equation under the assumption that $V = 0$ and $m = m_q = 0$ inside the volume and $m = m_q \to \infty$ outside.

1. First solve the Dirac equation for $V = 0$ and a finite value for $m$. There is no current outside the sphere.

2. Consider the parity of your ground state solutions. Conventionally the negative parity solution is identified with an s-wave ($S_{1/2}$), and the positive parity solution with a p-wave ($P_{1/2}$).

3. Consider as boundary condition that there is no quark current outside the hadron:

$$ n_\mu \bar{\psi} \gamma^\mu \psi |_{\partial V} = 0 $$

where $n_\mu$ is the 4-dimensional unit vector pointing away from the center of the sphere.

*Hint:* Consider

$$ -i \gamma^\mu n_\mu \psi |_{\partial V} = \psi |_{\partial V} $$

for your proof.

4. Determine $E$ and $k$ for $S_{1/2}$ and $P_{1/2}$ from the boundary condition.

5. Calculate the norm of your two solutions.

6. Suppose it requires energy to ‘make’ the sphere in which the quarks can move freely, so that the total energy of $n$ non-interacting quarks inside a volume $R$ is

$$ E_R = \frac{n x_0}{R} + \frac{4\pi}{3} R^3 B $$
where for the ground state $x_0 = 2.04$. $B$ is the energy density of the empty volume. Minimize the energy with respect to $R$ and show that

$$ R_{\text{min}} = \left( \frac{nx_0}{4\pi B} \right)^{1/4} $$
$$ E_{\text{min}} = \frac{4}{3} \left( n^3 x_0^3 4\pi B \right)^{1/4} $$

The proton consists of 3 quarks. What is its radius of $B^{1/4} \approx 100$ MeV?

II. Consider the Lagrangian density

$$ \mathcal{L} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial z} \right)^2 - m^2 \phi^2 \right] $$

where $\phi \equiv \phi(z,t)$ is a generalized coordinate.

1. Find the conjugate momentum $\pi(z,t)$ to $\phi$.
2. Find the equations of motion for the field and the solutions. Use periodic boundary conditions.
3. Suppose the field is expanded in normal modes

$$ \phi(z,t) = \sum_n c_n (a_n \varphi_n(z,t) + a_n^\dagger \varphi_n^*(z,t)), $$

where $a_n$ satisfy the commutation relations

$$ [a_n, a_{n'}] = [a_n^\dagger, a_{n'}^\dagger] = 0 $$
$$ [a_n, a_{n'}^\dagger] = \delta_{nn'} $$

(1)

Find the coefficients $c_n$ which will insure that the commutation relations assume the standard form

$$ [\phi(z,t), \pi(z',t)] = i\delta(z-z') $$

4. Find the Hamiltonian density, and express the Hamiltonian in terms of the number operator $a_n^\dagger a_n$.
5. What is the physical significance of this field?

Each sub-item is worth 10 points