Phys. 735: Homework III

due September 29, 2006

1. (5 pts)
Define a set of mappings $\mathcal{L}: M^4 \to M^4$: $\forall L \in \mathcal{L}$, there is $g = LgL^T$
where $M^4$ is Minkovski space with following metric $g$

$$g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

Prove that $\mathcal{L}$ forms a group.

2. (5pts)
Write the Lorentz transformation in terms of rapidity $R$:

$$\begin{pmatrix}
x'_0 \\
x'_1
\end{pmatrix} = \begin{pmatrix}
\cosh R & -\sinh R \\
-\sinh R & \cosh R
\end{pmatrix} \begin{pmatrix}
x_0 \\
x_1
\end{pmatrix}$$

where $\gamma = \cosh R$ and $\beta = \tanh R$.
Prove that

$$exp(-R\sigma_1) = \begin{pmatrix}
\cosh R & -\sinh R \\
-\sinh R & \cosh R
\end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.$$

Hint: Refer the related matrix algebra for calculating $e^A$, with $A$ being a matrix

3. (10 pts)
3.1 (4 pts)
Prove that in $M^4$, the boost along the x direction $L_x$ can be written in terms of the exponentiation of generator $L_1$:

$$L_x = exp(-R_xL_1)$$
where $R_x = \tanh(v_x/c)$ and $L_1$ is the extension of $\sigma_1$ written as

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2 (3 pts)
By the example of 3.1, find the generators $L_2$ and $L_3$ which are also natural extension of $\sigma_1$, which can generate the boost along $y$ and $z$ direction respectively via exponentiation.

3.3 (3 pts)
Verify that the commutator $[L_1, L_2]$ can generate a rotation along a certain direction and in a certain plane via exponentiation. Find the direction of rotation and the plane.