Phys. 735: Homework IV

due October 11, 2006

1. The minimum coupling of the electromagnetic field is written in a four-dimensional way as

\[ \hat{p}^\mu \to \hat{p}^\mu - \frac{e}{c} A^\mu \]

With this, the Klein-Gordon equation with an electromagnetic field is given as

\[ (\hat{p}^\mu - \frac{e}{c} A^\mu)(\hat{p}_\mu - \frac{e}{c} A_\mu)\psi = mc^2 \psi \]

(a) [5 pts]
Derive an expression for the four-current density in the electromagnetic field \( A_\nu \) and give expressions for the charge density and the charge-current density.

Then, consider a \( \pi^- \) meson (with mass \( m_\pi c^2 = 139.577 \) MeV and spin 0) being bound by the Coulomb potential

\[ V(r) = -\frac{Ze^2}{r} \]

in a stationary state of total energy \( E < m \). A stationary state of the Klein-Gordon equation has the form

\[ \Psi(r, t) = \psi(r) \exp(-iEt/\hbar) \]

where \( |E| \) is the energy per particle.

(b) [2 pts]
What is the time-independent Klein-Gordon equation for this potential?

(c) [4 pts]
Assume the radial and angular parts of the wave function \( \psi(r) \) separate. Verify that this yields

\[ \frac{d^2 u_l(kr)}{dr^2} + \left[-\frac{2EZ\alpha}{r} - \frac{(m^2 - E^2)}{r^2} - \frac{l(l + 1) - (Z\alpha)^2}{r^2}\right] u_l(kr) = 0 \]

where

\[ \alpha = e^2 \equiv \frac{\epsilon^2}{\hbar c} \approx \frac{1}{137} \]

(d) [3 pts]
Show that this equation can be written in the dimensionless form

\[ \left[ \frac{d^2}{d\rho^2} - \frac{\mu^2 - \frac{1}{4}}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4} \right] u_l(\rho) = 0 \]
with

\[
\begin{align*}
\rho &= \gamma^r \\
\gamma^2 &= 4(m^2 - E^2) \\
\mu^2 &= \left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2 \\
\lambda &= \frac{2EZ\alpha}{\gamma}
\end{align*}
\]

(e) [3 pts]
Assume this equation has a solution in the usual form of a power series times the \( \rho \to \infty \) and \( \rho \to 0 \) solutions:

\[ u_l(\rho) = \rho^k (1 + c_1 \rho + c_2 \rho^2 + c_3 \rho^3 + \cdots) e^{-\rho/2} \]  

Show that

\[ k = k_\pm = \frac{1}{2} \pm \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2} \]

(f) [4 pts]
Show that for both \( k_+ \) and \( k_- \) the wave function is divergent at the origin yet normalizable.

(g) [4 pts]
Show that only for \( k_+ \) is the expectation value of the kinetic energy finite:

\[ \int dr \; r^2 \left[ \frac{d(u_l/r)}{dr} \right]^2 < \infty \]

(h) [4 pts]
Show that the \( k_+ \) solution has a nonrelativistic limit which agrees with the solution found for the Schrödinger equation.

(i) [4 pts]
Determine the recurrence relation among the \( c_i \)'s for this to be a solution of the Klein-Gordon equation.

(j) [3 pts]
Show that unless the power series of Eq. 1 terminates, the wave function will have an incorrect asymptotic form.
(k) [4 pts]
Show that the termination condition determines the eigen energy for the $k_+$ solution to be

$$E = m \left( 1 + (Z\alpha)^2 \left[ n - l - \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2} \right]^2 \right)^{1/2}$$

where $n$ is the principal quantum number.

(l) [4 pts]
Expand $E$ in powers of $\alpha^2$ and show that the $\alpha^2$ term yields the Bohr formula, and that higher order terms can be identified with relativistic corrections.

(m) [3 pts]
Is the $l$-degeneracy present in the nonrelativistic theory now removed? (And if so, to what order in $\alpha$?)