Drag Forces on Slender Rods

• Force applied results in translation, rotation, bending or buckling

• Speed of motion determined by Hydrodynamic Properties

• Drag Forces and their effects
Table 6.2 Drag coefficients in an unbounded solution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Direction</th>
<th>Cylinder (L&gt;&gt;r)</th>
<th>Ellipsoid (b&gt;&gt;a)</th>
<th>Sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = \gamma_{</td>
<td></td>
<td>}v$</td>
<td></td>
<td>$\frac{2\pi\eta L}{\ln(L/2r) - 0.20}$</td>
</tr>
<tr>
<td>$T = \gamma_{r}a\omega$</td>
<td></td>
<td>$\frac{1}{3}\pi\eta L^3}{\ln(L/2r) - 0.66}$</td>
<td>$\frac{8\pi\eta b^3}{\ln(2b/a) - 0.5}$</td>
<td>$8\pi\eta r^3$</td>
</tr>
</tbody>
</table>

Note: The parallel, $\gamma_{||}$, and perpendicular, $\gamma_{\perp}$, drag coefficients are defined by $F = \gamma_{||}v$ and $F = \gamma_{\perp}v$, where $F$ is the force and $v$ is the velocity. The rotational drag coefficients are defined by $T = \gamma_{r}\omega$ and $T = \gamma_{a}\omega$, where $T$ is the torque and $\omega$ is the angular velocity. The values for cylinders are from Tiraldo and García de la Torre, 1981, and those for prolate ellipsoids are from Perrin, 1934.
Example 6.5 Drag on a sperm  A human sperm has a head of length 5.8 μm, a width of 3.1 μm, and a tail of length $L = 36$ μm, which tapers from a diameter of $\sim 1$ μm down to $\sim 0.16$ μm (see figure below). The ratio of the volumes of the tail and the head is $\sim 0.3$. Yet for movement parallel to the tail’s axis, the ratio of the drag coefficients is $\equiv 1.5$. This illustrates the rule of thumb that the drag on an object is determined primarily by the largest dimension, even when motion is parallel to the long axis. For a sperm moving at $v = 50$ μm/s, the drag force given by $\sim (\gamma_{\text{head}} + \gamma_{\text{tail}}) \cdot v \equiv 5$ pN. More exact formulas for the drag coefficients of complex shapes can be found in Garcia de la Torre and Bloomfield (1981).
### Table 6.3 Drag coefficients (per unit length) for a cylinder near a plane surface

<table>
<thead>
<tr>
<th>Definitions</th>
<th>Drag coefficient</th>
<th>Force or torque</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_V$</td>
<td>$c_{</td>
<td></td>
</tr>
<tr>
<td>$c_{\perp}$</td>
<td>$c_{\perp} = 2c_{</td>
<td></td>
</tr>
<tr>
<td>$c_v$</td>
<td>$c_v = \frac{1}{(c_{\perp}^{-1} - c_a^{-1})}$</td>
<td>$F = c_v L v$</td>
</tr>
<tr>
<td>$c_a$</td>
<td>$c_a = \frac{4\pi \eta}{\left[1 - (r/h)^2\right]^{1/2}}$</td>
<td>$T = c_a L r^2 \omega$</td>
</tr>
<tr>
<td>$c_r$</td>
<td>$c_r = \frac{1}{3} c_{\perp}$</td>
<td>$T = c_r \omega (L_1^3 + L_2^3)$</td>
</tr>
</tbody>
</table>

**Source:** From Hunt et al., 1994.
**Example 6.6 Drag forces in gliding assays**  In an in vitro motility gliding assay (Chapter 13), a single kinesin molecule can move a microtubule of radius \( \sim 15 \) nm at a speed, \( v \), of \( \sim 1 \) \( \mu \)m/s, independent on the microtubule’s length (for lengths up to \( L = 20 \) \( \mu \)m). Thus the motor must be able to exert a force well in excess of the drag force \( F_{\text{drag}} = c_{||} L \cdot v = 2\pi \eta L v / \cosh^{-1}(h/r) \approx 0.1 \) pN, assuming that the axis of the microtubule is height \( h = 25 \) nm above the surface. Indeed, to appreciably slow the movement of a 10-\( \mu \)m-long microtubule (to 0.25 \( \mu \)m/s), the viscosity must be increased 100-fold (Hunt et al., 1994). Using this approach, Hunt et al. (1994) deduced that a kinesin molecule could exert forces up to \( \sim 5 \) pN against a viscous load.
Dynamics of Bending and Buckling

• Bending of filament gives rise to motion perpendicular to filament

![Diagram showing bending and drag force]

• At each portion along length of the rod, Drag force/length = 
  \[ f_\perp(x) = -c_\perp v_\perp(x) = -c_\perp \frac{\partial y}{\partial t} \]
Total bending moment at point $x$ due to drag forces

$$M(x) = \int_{x}^{L} f_{\perp}(x')(x'-x).dx'$$

Differentiating twice

$$\frac{\partial^2 M}{\partial x^2} = f_{\perp}(x)$$

$$\frac{\partial^4 y}{\partial x^4} = -\frac{c_{\perp}}{EI} \frac{\partial^2 y}{\partial t}$$

This is referred to as the Hydrodynamic Beam equation

- This is solved by using separation of variables

$$f_{\perp}(x) = -c_{\perp} v_{\perp}(x) = -c_{\perp} \frac{\partial y}{\partial t}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI}$$
• How long does it take for a bent rod to relax to straight position?

• Solving Hydrodynamic beam equations with appropriate boundary conditions

\[
\frac{\partial^4 y}{\partial x^4} = -\frac{c_{\perp}}{EI} \frac{\partial y}{\partial t}
\]

Boundary conditions

\[
\left. \frac{\partial^3 y}{\partial x^3} \right|_{x=0} = \left. \frac{\partial^3 y}{\partial x^3} \right|_{x=L} = \left. \frac{\partial^2 y}{\partial x^2} \right|_{x=0} = \left. \frac{\partial^2 y}{\partial x^2} \right|_{x=L} = 0
\]
Solution

**n odd**

\[ y_n(x,t) = e^{-t/\tau_n} \left[ \sinh \alpha_n \cos \frac{2\alpha_n}{L} \left( x - \frac{L}{2} \right) - \sin \alpha_n \cosh \frac{2\alpha_n}{L} \left( x - \frac{L}{2} \right) \right] \]

**n even**

\[ y_n(x,t) = e^{-t/\tau_n} \left[ \cosh \alpha_n \sin \frac{2\alpha_n}{L} \left( x - \frac{L}{2} \right) + \cos \alpha_n \sinh \frac{2\alpha_n}{L} \left( x - \frac{L}{2} \right) \right] \]

**amplitude** \( = e^{-t/\tau_n} \)

\[ \tau_n \cong \frac{c_\perp}{EI} \left( \frac{L}{\pi(n + 1/2)} \right)^4 \]

\( n = 1,2,3,\ldots \)
Example 6.7 Relaxation of microtubules and actin filaments

For a microtubule of length 50 μm and radius 15 nm, of flexural rigidity $30 \times 10^{-24}$ N·m², and height 1000 nm above a surface immersed in an aqueous solution (viscosity 1 mPa·s), the relaxation time for the first mode is 1.1 s, and that for the second mode is 0.14 s. A microtubule of length 5 μm has a relaxation time of $\sim 100 \mu$s, $\sim 10,000$ times smaller. Actin filaments relax considerably more slowly due to their greater flexibility: With a flexural rigidity of $\sim 60 \times 10^{-27}$ N·m² (see Table 6.4 and Equation 6.11), a 50-μm-long actin filament will have a first-mode relaxation time of $\sim 500$ s (ignoring the small difference in drag coefficients due to the different radii of the filaments) (Gittes et al., 1993).
Thermal Bending of Filaments

- Thermal forces cause fluctuation in shape of filament
- DNA or unfolded chain of amino acids become highly unstructured

- Straight actin filament or ordered set of amino acid must be rigid to maintain their structure

- How rigid? And what is the flexural rigidity?
Persistence Length

- Filament’s resistance to thermal forces governed by persistence length

\[
\langle \cos[\theta(s) - \theta(0)] \rangle = \exp \left( - \frac{s}{2L_p} \right)
\]

For 2D filament,

\[
L_p = \frac{EI}{kT}
\]
If,
Length of filament > persistence length
Then, tangent angles are not correlated
Thermal Bending of Semi flexible Polymers

• For semi flexible polymers,

Original length = persistence length

Persistence length can be calculated by using mean squared end to end length

\[
\langle R^2 \rangle = 2L_p^2 \left[ \exp \left( -\frac{L}{L_p} \right) - 1 + \frac{L}{L_p} \right]
\]

for \( L << L_p \) \hspace{1cm} \text{for } L >> L_p

\[
\langle R^2 \rangle = 0 \hspace{1cm} \langle R^2 \rangle = 2L_pL
\]
Entropic Elasticity of a Freely Jointed Chain

Mechanics of proteins having segmental flexibility which have Globular or rigid coil domains Linked by flexible regions

\[ \langle X \rangle = nb \langle \cos \theta \rangle \]
• Force extension curve: measured by determining $F$, which extends the chain by $\langle X \rangle$

$$\langle X \rangle = nb.L\left(\frac{Fb}{kT}\right)$$

Where, $L\left(\frac{Fb}{kT}\right)$ is the Langevin function

$$L\left(\frac{Fb}{kT}\right) = \frac{e^{\frac{Fb}{kT}} + e^{-\frac{Fb}{kT}}}{e^{\frac{Fb}{kT}} - e^{-\frac{Fb}{kT}}} - \frac{1}{\left(\frac{Fb}{kT}\right)}$$
• For small values of $F$, 
\[
L\left(\frac{Fb}{kT}\right) \approx \frac{1}{3} \frac{Fb}{kT}
\]

\[
\left< X \right> = nb \cdot \frac{1}{3} \frac{Fb}{kT}
\]

\[
F = \frac{3kT}{nb^2} \left< X \right>
\]

which compares to $F = kx$ (Force applied on a spring)

Where, $k = \frac{3kT}{nb^2}$
Worm Like Chain

- Length of rod > persistence length
- For freely jointed chain, \( \langle R^2 \rangle = nb^2 \)
- For slender rods in three dimensions, \( L \gg L_p \)
  \[
  \langle R^2 \rangle \approx 2LL_p \quad nb = L
  \]
  \[
  2L_p = b
  \]
  which is called Kuhn length
- In the absence of force, worm like chain is equal to freely jointed chain whose segment length \( b \) is twice the persistence length \( (L_p) \)
Summary

• Resistance of slender rods to bending forces defined by flexural rigidity (EI)
• If bending moments are known, shape can be calculated by beam equation
• If rod is in a fluid, bending opposed by drag forces
• Flexible rods are also bent by thermal forces
• Persistence length defines thermal fluctuations
• Models like freely jointed chain and worm like chain