Chapter 10
Scattering by a Central Potential

Abstract: With the central role played by scattering in any measurements on a quantum system, in this chapter we introduce the concepts in scattering theory with application to the simple square well problem as well as the Coulomb problem.

Most scattering experiments consist of a beam of particles incident on a stationary target, and a detector to measure the number of scattered particles in a given direction per unit time. In practice, the beam has many particles in it, but we will neglect any interaction between the particles. Similarly, the target has many particles in a confined region of space and again we will assume the interaction between the target particles can be ignored. Although we should be considering a wave packet incident on the target, it can be shown\(^1\) that we can get the same result for the scattering amplitude assuming the incident beam is described by a plane wave. In this way we reduce the complexity of the algebra in the formulation of the scattering problem. The plane incident wave with momentum \(\mathbf{k}_i\) is then given, up to a normalization, by

\[
\psi_{\text{inc}}(\mathbf{r}) \propto e^{i \mathbf{k}_i \cdot \mathbf{r}}.
\]

On the other hand, the scattered particles will be originating at the target and moving in a spherical wave. The amplitude of this scattered wave might depend on the scattering angles. However, because of the cylindrical symmetry about the incident beam, the only angle dependence is the angle between the incident beam direction, \(\hat{k}_i\), and the scattered particle’s direction of momentum \(\hat{k}_f\). Here, the subscripts \(i\) and \(f\) refer to the initial and final states. We now can write the scattered wave in terms of a spherical wave, \(e^{ikr}/r\), as

\[
\psi_{\text{scat}} \propto f(k_i, \hat{k}_i \cdot \hat{k}_f) \frac{e^{ik_f r}}{r}.
\]

\(^1\)For a full treatment of scattering using wave packets consult Goldberger and Watson [28].
For elastic scattering, i.e. when the initial energy of the incident particle is equal to the final energy of the scattered particle, we have that $|\vec{k}_i| = |\vec{k}_f| = k$, and we can write the time independent wave function outside the interaction region as

$$\psi(\vec{r}) \propto e^{i\vec{k}_i \cdot \vec{r}} + f(k, \theta) \frac{e^{ikr}}{r},$$

(10.3)

where $\cos \theta = \hat{k}_i \cdot \hat{k}_f$ and $f(k, \theta)$ is the amplitude of the scattered wave or the \textit{scattering amplitude}. This wave function is illustrated in Figure 10.1.

![Figure 10.1](image)

**Figure 10.1**: Illustration of the asymptotic wave function for a scattering problem. It consists of an incident plane wave and an outgoing spherical wave to represent the scattered particles.

In the next two sections we will relate the probability for scattering in a given direction to the coefficient of the outgoing spherical wave $f(k, \theta)$. We will then relate the amplitude $f(k, \theta)$ to the solution of the Schrödinger equation. In this way we establish the relation between the wave function and the quantities we measure in any scattering experiment. In the final section of this chapter we will consider the problem of scattering by a Coulomb potential, which will require special treatment because the interaction in this case has an infinite range.

### 10.1 The Cross Section

In any scattering experiment, we measure the number of particles scattered at a given angle $\theta$. If we divide the number of scattered particles per unit time by the flux of incident particles, we get the probability for a particle scattering at a given angle. This is referred

\footnote{This definition of elastic scattering assumes we are in the center of mass where $\vec{k}_i$ and $\vec{k}_f$ are the relative initial and final momentum (see Sec. 2.2).}
to as the differential cross section for that angle. In this section we derive the relation
between the cross section and the scattering amplitude $f(k, \theta)$.

The time dependent Schrödinger equation for the potential $V(r)$ is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2\mu} \nabla^2 + V(r)\right) \Psi(r, t) ,$$

where $\mu$ is the reduced mass (see Eq. (8.12)) of the incident and target particles. The
complex conjugate solution, $\Psi^*(r, t)$ satisfies the equation

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = \left(-\frac{\hbar^2}{2\mu} \nabla^2 + V(r)\right) \Psi^*(r, t) .$$

Multiplying Eq. (10.4) by $\Psi^*$ from the left, and Eq. (10.5) by $\Psi$ from the right and
subtracting, we get

$$i\hbar \frac{\partial}{\partial t} (\Psi^* \Psi) = -\frac{\hbar^2}{2\mu} \left( \Psi^* \nabla^2 \Psi - (\nabla^2 \Psi^*) \Psi \right)$$

$$= -\nabla \cdot \left( \frac{\hbar^2}{2\mu} (\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi) \right) .$$

(10.6)

If we now define the density, $\rho = \Psi^* \Psi$, then Eq. (10.6) can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 ,$$

(10.7)

where the current $\vec{j}$ is given by

$$\vec{j} = \frac{\hbar}{2\mu} \left( \Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi \right)$$

$$= \frac{\hbar}{2\mu} \left( \psi^* \nabla \psi - (\nabla \psi^*) \psi \right) .$$

(10.8)

Here, we have dropped the time dependence in the wave function since we are not dealing
with wave packets. In any scattering experiment the scattered particle is measured at a
distance $r$ which is large compared to the range of the interaction. This means that the
detector is in the asymptotic region where the wave function is given by

$$\psi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} + f(k, \theta) \frac{e^{ikr}}{r} .$$

(10.9)

Using Eq. (10.9) in Eq. (10.8), we get, after some algebra that involves writing $\nabla$ in
spherical polar coordinates,\(^3\) an expression of the form

\[
\vec{J} = \frac{\hbar \hat{k}_i}{\mu} + \frac{\hbar k}{\mu} \frac{\hat{e}_r}{r^2} |f(k, \theta)|^2 \\
+ e^{ikr(1 - \cos \theta)} [\ldots] + e^{-ikr(1 - \cos \theta)} [\ldots] .
\]

(10.10)

Since the detector is never put in the forward direction, i.e. \(\theta = 0\), and it subtends a finite solid angle, we need to integrate over the angular range of the detector, i.e.,

\[
\int d\theta d\phi \sin \theta \ g(\theta, \phi) \ e^{\pm ikr(1 - \cos \theta)} ,
\]

where \(g(\theta, \phi)\) is a smooth function that depends on the detector’s properties. For \(kr \gg 1\), we are integrating a highly oscillatory function, and the integral is zero according to the Riemann-Labegue lemma. We thus have for the current

\[
\vec{J} = \frac{\hbar \hat{k}_i}{\mu} + \frac{\hbar k}{\mu} \frac{\hat{e}_r}{r^2} |f(k, \theta)|^2 .
\]

(10.11)

The first term represents the incident beam of particles, and is present even if there was no scattering target. The radial flux of scattered particles is then given by

\[
\vec{J} \cdot \hat{e}_r = \frac{\hbar k}{\mu} \frac{|f(k, \theta)|}{r^2} .
\]

(10.12)

![Figure 10.2: Illustration of the scattering angle \(\theta\), and the solid angle \(d\Omega\), subtended by the detector \(D\).](image)

The number of particles crossing the area that subtends a solid angle \(d\Omega\) is given by (see Figure 10.2)

\[
\vec{J} \cdot \hat{e}_r \ dA = \frac{\hbar k}{\mu} \frac{|f(k, \theta)|^2}{2} \ d\Omega .
\]

(10.13)

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\(^3\)In spherical polar coordinates we have

\[
\nabla = \hat{e}_r \ \frac{\partial}{\partial r} + \hat{e}_\theta \ \frac{1}{r} \ \frac{\partial}{\partial \theta} + \hat{e}_\phi \ \frac{1}{r \sin \theta} \ \frac{\partial}{\partial \phi} .
\]
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The differential cross section is the flux of scattered particles per unit area divided by the incident flux, $\hbar k/\mu$. Therefore we have that

$$d\sigma = |f(k, \theta)|^2 \ d\Omega$$

and we have

$$\frac{d\sigma}{d\Omega} = |f(k, \theta)|^2 . \quad (10.14)$$

This in fact is the result we would have expected from the construction of Eq. (10.3) for the asymptotic wave function. We now have to relate the scattering amplitude $f(k, \theta)$ to the solution of the Schrödinger equation. One important observation we can make at this stage is that the cross section measured experimentally depends on the form of the wave function for large $r$, i.e., outside the interaction region.

10.2 Kinematics

Consider the scattering of two particles where the potential between the particles is a function of the relative distance between the particles, i.e. $V(|\vec{r}_1 - \vec{r}_2|)$. The Hamiltonian for such a system is given by

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|) , \quad (10.15)$$

where $p_1^2 = -\hbar^2 \nabla_1^2$ and $p_2^2 = -\hbar^2 \nabla_2^2$. We introduce relative and center of mass coordinates and momenta (see Eqs. (8.6) and (8.7)), i.e.

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{p} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{m_1 + m_2} \quad \text{and} \quad \vec{P} = \vec{p}_1 + \vec{p}_2 . \quad (10.16)$$

In terms of these new variables the Hamiltonian takes the form

$$H = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(r) , \quad (10.17)$$

where

$$M = m_1 + m_2 \quad \text{and} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} . \quad (10.18)$$

Here, $\mu$ is the reduced mass, and $M$ the total mass. In the center of mass we have $\vec{P} = 0$, and the two-particle Hamiltonian reduces to the one-particle Hamiltonian with a reduced mass $\mu$, i.e.

$$H = \frac{\vec{p}^2}{2\mu} + V(r) . \quad (10.19)$$
We thus have reduced the two-particle problem to a one-particle problem of mass $\mu$ scattering from a potential $V(r)$. The Schrödinger equation in the two-particle center of mass is

$$\left( -\frac{\hbar^2}{2\mu}\nabla^2 + V(r) \right) \psi(\vec{r}) = E\psi(\vec{r}) .$$  \hfill (10.20)

### 10.3 The Square Well Potential

Before we proceed to a general discussion of the solution of Eq. (10.20), let us consider the solution of Eq. (10.20) for the case of a simple square well of radius $a$, i.e.,

$$V(r) = \begin{cases} 
-V_0 & \text{for } r < a \\
0 & \text{for } r > a 
\end{cases} .$$  \hfill (10.21)

In the last chapter, we considered the bound state (i.e. $E < 0$) problem for this potential. We now have to consider the case of $E > 0$, i.e. the scattering problem. For $r < a$, the solution of the radial equation is the same as for the bound state (see Eq. (9.19)), which is

$$R_\ell(r) = A \, j_\ell(\alpha r) \quad \text{with} \quad \alpha^2 = \frac{2\mu}{\hbar^2} (E + V_0) \quad \text{for} \quad r < a .$$  \hfill (10.22)

For $r > a$, we have a linear combination of the two solutions of the radial equation, i.e.,

$$R_\ell(r) = B \, j_\ell(kr) + C \, n_\ell(kr) ,$$  \hfill (10.23)

or

$$R_\ell(r) = B' \, h_\ell^{(+)}(kr) + C' \, h_\ell^{(-)}(kr) ,$$  \hfill (10.24)

where $k^2 = 2\mu E/\hbar^2$. Since we are not considering bound states, the wave function $R_\ell(r)$ does not go to zero as $r \to \infty$, otherwise, the current at the detector would be zero. This means neither $B$ nor $C$ ($B'$ or $C'$) is zero. In this case either combination is valid and we will use both at different times.

To get a feeling for the form of the scattering wave function, let us take $\ell = 0$. In this case we have that

$$R_0(r) = \frac{u_0(r)}{r} ,$$  \hfill (10.25)

with

$$u_0(r) = \begin{cases} 
A \sin \alpha r & \text{for } r < a \\
B \sin kr + C \cos kr & \text{for } r > a 
\end{cases} .$$  \hfill (10.26)

An alternative way of writing the solution for $r > a$ is

$$u_0(r) = A' \sin(kr + \delta) \quad \text{for} \quad r > a .$$  \hfill (10.27)
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Note that both forms for the solution for $r > a$ have two constants to be adjusted by the boundary condition which requires that the wave function and its derivative be continuous at $r = a$. Before we determine these constants, let us examine the form of the wave function for the case of $V_0 = 0$, and $V_0 \neq 0$ at large distances, i.e. for $r > a$. In Figure 10.3, we sketch both wave functions. We observe that for $V_0 \neq 0$ the wave length for $r < a$ is smaller than is the case for $r > a$. However, for $V_0 = 0$, the wave length is the same for all $r$. Thus the presence of the potential, shifts the wave function for $r > a$ relative to the wave function for the case $V_0 = 0$ by an amount $\delta$. In other words, the constant $\delta$ introduced in Eq. (10.27) depends on the parameters of the potential, in this case $V_0$ and $a$. The inverse might also be possible, i.e., if we measure $\delta$ we might be able to determine the parameters of the potential. The $\delta$ introduced in Eq. (10.27) is called the scattering Phase Shift.

To guarantee that the wave function and its derivative are continuous at $r = a$, we take

$$A \sin \alpha a = A' \sin(ka + \delta)$$
$$\alpha A \cos \alpha a = kA' \cos(ka + \delta).$$

Therefore, to determine $\delta$, we take the ratio of the above two equations. This gives us the result that

$$\alpha \cot \alpha a = k \cot(ka + \delta)$$
$$= k \frac{\cot ka \cot \delta - 1}{\cot \delta + \cot ka}. \quad (10.28)$$

Solving this equation for $\cot \delta$ we get

$$k \cot \delta = \alpha \cot \alpha a + k \tan ka \frac{1 - \frac{\alpha}{k} \cot \alpha a \tan ka}{1 - \frac{\alpha}{k} \cot \alpha a \tan ka}. \quad (10.29)$$
Having determined the phase shift $\delta$ we can determine one of the other two constants ($A$ or $A'$). This leaves one overall multiplicative constant to be determined. For bound states, this constant was determined by the requirement that the wave function be normalized. However, the scattering wave function is not normalizable in the same manner as the bound state, because the normalization integral is mathematically not well defined.\footnote{We will show in a later chapter that the scattering wave function has a $\delta$-function normalization.} This is a consequence of the fact that the wave function does not go to zero as $r \to \infty$.

![Figure 10.4: The scattering plane in the two-body center of mass. The incident beam is along the positive z-axis.](image)

### 10.4 The Scattering Amplitude

Having considered the simple problem of S-wave (i.e. $\ell = 0$) scattering by a square well, we now turn to the more general problem of two-particle scattering. In the two-body center of mass, we have two particles with opposite momenta, initially along the $z$-axis (see Figure 10.4). Since the scattering takes place in a plane, we can eliminate the $\phi$ dependence in this problem. Furthermore, the initial momentum defines a direction in space, thus braking the isotropy of the space. This implies that the angular momentum of the system is not fixed as was the case for the bound state problem, but depends on the momentum in the initial state. In fact, the angular momentum, classically, is perpendicular to the scattering plane. In general, the wave function is a linear combination of many such angular momenta. As we have chosen our $z$-axis to be along the direction of the incident momentum and therefore in the scattering plane, we expect the projection of our angular momentum along the $z$-axis to be zero. This means that the wave function for a given angular momentum is of the form

$$R_\ell(r) Y_{\ell 0}(\theta, \phi) .$$

But

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos \theta) ,$$

(10.31)
where \( \cos \theta = \hat{r} \cdot \hat{k} \). We now can write the general form of the wave function as

\[
\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\ell} i^\ell (2\ell + 1) \psi_\ell(r) P_\ell(\cos \theta).
\] (10.32)

The choice of the factor of \((2\pi)^{-3/2} i^\ell (2\ell + 1)\) is for later convenience in the normalization. At this stage we would like to point out that this choice for the normalization is not unique.

Since \(P_\ell(\cos \theta)\) is related to the spherical harmonics, it is an eigenstate of the angular momentum operator square, \(L^2\). Making use of the orthogonality of \(P_\ell\), we can show that the radial Schrödinger equation for \(\psi_\ell(r)\) is given by

\[
\frac{d^2 \psi_\ell}{dr^2} + \frac{2}{r} \frac{d \psi_\ell}{dr} - \frac{\ell(\ell + 1)}{r^2} \psi_\ell + [k^2 - U(r)]\psi_\ell(r) = 0,
\] (10.33)

where

\[
k^2 = \frac{2\mu E}{\hbar^2} \quad \text{and} \quad U(r) = \frac{2\mu}{\hbar^2} V(r).
\] (10.34)

We now assume that the potential satisfies the condition that \(U(r) \to 0\) faster than \(r^{-1}\). In this case the radial equation for large \(r\) is given by

\[
\frac{d^2 \psi_\ell}{dr^2} + \frac{2}{r} \frac{d \psi_\ell}{dr} - \frac{\ell(\ell + 1)}{r^2} \psi_\ell(r) + k^2 \psi_\ell(r) = 0
\] (10.35)

which is the spherical Bessel’s equation.

**Note:** Our assumption that \(U(r) \to 0\) faster than \(r^{-1}\) excludes the Coulomb potential. From this point on we restrict ourselves to the class of potentials that satisfy the above condition. We will examine the Coulomb potential as a special case at the end of this chapter.

Before we write the general solution to Eq. (10.33), let us rewrite the solution to the square well potential in terms of the spherical Bessel function. We have from Eq. (10.25) and (10.27) that, for \(\ell = 0\),

\[
\psi_0(r) = \frac{A'}{r} \sin(kr + \delta) = \frac{A'}{r} \left[ \sin kr \cos \delta + \cos kr \sin \delta \right] = kr A' \left[ \cos \delta j_0(kr) + \sin \delta n_0(kr) \right].
\] (10.36)

This gives us the idea of writing the general solution to Eq. (10.35), for any angular momentum, \(\ell\), as

\[
\psi_\ell(r) \to A_\ell \left[ \cos \delta_\ell j_\ell(kr) + \sin \delta_\ell n_\ell(kr) \right] \quad \text{for} \quad r \to \infty.
\] (10.37)
Since for \( r \to \infty \) we have that (see Eq. (9.12))

\[
\begin{align*}
\hat{j}_\ell(\rho) & \to \frac{1}{\rho} \sin \left( \rho - \frac{\pi \ell}{2} \right) \\
n_\ell(\rho) & \to \frac{1}{\rho} \cos \left( \rho - \frac{\pi \ell}{2} \right),
\end{align*}
\]

then

\[
\psi_\ell(r) \to \frac{A_\ell}{kr} \sin \left( kr - \frac{\pi \ell}{2} + \delta_\ell \right) \text{ for } r \to \infty .
\]

Here, \( A_\ell \) and \( \delta_\ell \) are the two constants to be determined by the normalization of the wave function and the continuity of the logarithmic derivative.\(^5\) We now would like to make use of Eqs. (10.39) and (10.32) to write the total asymptotic wave function in the form given by Eq. (10.3). In this way we hope to relate the scattering amplitude \( f(k, \theta) \) to the phase shifts \( \delta_\ell \). To achieve this result we first write \( \sin(kr - \pi \ell/2 + \delta_\ell) \) in terms of exponentials, to get

\[
\psi_\ell(r) \to \frac{A_\ell}{kr} \frac{1}{2i} \left[ e^{i(kr - \frac{\pi \ell}{2} + \delta_\ell)} - e^{-i(kr - \frac{\pi \ell}{2} + \delta_\ell)} \right] = \frac{A_\ell}{2ikr} e^{-i\delta_\ell} \left[ -e^{-i(kr - \frac{\pi \ell}{2})} + e^{2i\delta_\ell} e^{i(kr - \frac{\pi \ell}{2})} \right] = \frac{A_\ell}{2ikr} e^{-i\delta_\ell} \left[ 2i \sin \left( kr - \frac{\pi \ell}{2} \right) + (e^{2i\delta_\ell} - 1) e^{i(kr - \frac{\pi \ell}{2})} \right].
\]

Using Eq. (10.39) we get the asymptotic radial wave function for a given \( \ell \) to be of the form

\[
\psi_\ell(r) \to A_\ell e^{-i\delta_\ell} \left( \hat{j}_\ell(kr) + (-i) \frac{1}{2ik} \left( e^{2i\delta_\ell} - 1 \right) e^{ikr} \right). \quad (10.41)
\]

Using this result in Eq. (10.32), and the fact that \( A_\ell = e^{i\delta_\ell} \), we get the scattering wave function for \( r \to \infty \) to be

\[
\psi(\vec{r}) \to \frac{1}{(2\pi)^{3/2}} \left( e^{i\vec{k} \cdot \vec{r}} + f(k, \theta) \frac{e^{ikr}}{r} \right), \quad (10.42)
\]

\(^5\)The logarithmic derivative is given by

\[
\frac{d}{dr} \log \psi_\ell(r) = \frac{1}{\psi_\ell(r)} \frac{d\psi_\ell(r)}{dr}
\]

which is identical to the condition given in Eq. (10.28) for the square well with \( \ell = 0 \).
where
\[ e^{i\vec{k} \cdot \vec{r}} = \sum_{\ell} i^\ell (2\ell + 1) j_{\ell}(kr) P_\ell(\cos \theta) \, , \tag{10.43} \]
and
\[ f(k, \theta) = \frac{1}{k} \sum_{\ell} (2\ell + 1) \left( \frac{e^{2i\delta_\ell} - 1}{2i} \right) P_\ell(\cos \theta) \, , \tag{10.44} \]
where \( \cos \theta = \hat{k} \cdot \hat{r} \). Now since the direction of the final momentum \( \vec{k}_f \) is the radial direction, i.e., \( \hat{r} = \hat{k}_f \), we have \( \cos \theta = \hat{k}_i \cdot \hat{k}_f \), with \( \vec{k}_i \) being the initial momentum of the particles in the beam.

In Eq. (10.44), we have established the relation between the scattering amplitude and the phase shifts. In this way we have completed the relation between the experimentally measured cross section and the wave function which is a solution to the Schrödinger equation. Finally, we can write Eq. (10.44) as
\[ f(k, \theta) = \frac{1}{k} \sum_{\ell} (2\ell + 1) f_\ell(k) P_\ell(\cos \theta) \, , \tag{10.45} \]
where \( f_\ell(k) \), the partial wave amplitude, is given by
\[ f_\ell(k) = \frac{1}{2i} \left( e^{2i\delta_\ell} - 1 \right) = e^{i\delta_\ell} \sin \delta_\ell \, . \tag{10.46} \]
In general, this partial wave amplitude is a complex number, i.e. it has a magnitude and a phase. With the help of Eq. (10.14) we can write the differential cross section in terms of the partial wave amplitude as
\[ \frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left| \sum_{\ell} (2\ell + 1) f_\ell(k) P_\ell(\cos \theta) \right|^2 \, . \tag{10.47} \]
This gives the probability for an incident particle, with momentum \( k \) and energy \( E = \frac{k^2}{2m} \), to be scattered in the direction defined by the angle \( \theta \). To get the total cross section, i.e. the probability of scattering in any direction, we have to integrate the differential cross section over the 4\( \pi \) solid angle, i.e.
\[ \sigma_T = \int \left( \frac{d\sigma}{d\Omega} \right) d\Omega = \frac{1}{k^2} \sum_{\ell} \sum_{\ell'} (2\ell + 1)(2\ell' + 1) f_\ell(k) f^*_\ell'(k) \int d\Omega P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \, . \]
Using the orthogonality of the Legendre polynomials (i.e. Eq. (8.60)), we get
\[ \sigma_T = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) |f_\ell(k)|^2 = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \sin^2 \delta_\ell \, . \tag{10.48} \]
Comparing Eqs. (10.47) and (10.48), we observe that the differential cross section has more information about the scattering amplitude than the total cross section. To illustrate this, consider the case when only two partial waves are important, the $\ell = 0$ and 1. In this case we have for the differential cross section

$$
\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left( |f_0|^2 + 9|f_1|^2 \cos^2 \theta + 3(f_0 f_1^* + f_0^* f_1) \cos \theta \right),
$$

while for the total cross section we have

$$\sigma_T = \frac{4\pi}{k^2} \left( |f_0|^2 + 3|f_1|^2 \right).$$

Thus we see that the differential cross section will give us information on the relative phase of the $\ell = 0$ and $\ell = 1$ amplitudes. Here, we note that in the event of $S$-wave scattering only, the differential cross section is angle independent, i.e. the differential cross section is isotropic.

### 10.5 The Optical Theorem

This theorem, is a special case of a more general theorem that sets a nonlinear constraint on the scattering amplitude called unitarity. In particular it gives a relation between the scattering amplitude in the forward direction, $f(k, \theta = 0)$, and the total cross section, $\sigma_T$. It results from the condition that probability should be conserved in the scattering process.

From Eq. (10.45), we have that the forward scattering amplitude is given by

$$f(k, \theta = 0) = \frac{1}{k} \sum_{\ell} (2\ell + 1) f_\ell(k) P_\ell(1)$$

$$= \frac{1}{k} \sum_{\ell} (2\ell + 1) f_\ell(k)$$

(10.51)

since $P_\ell(1) = 1$. Taking the imaginary part of this equation we get

$$\mathcal{I} \left[ f(k, 0) \right] = \frac{1}{k} \sum_{\ell} (2\ell + 1) \mathcal{I} \left[ f_\ell(k) \right].$$

(10.52)

But we have from Eq. (10.46) that for real phase shifts

$$\mathcal{I} \left[ f_\ell(k) \right] = \sin^2 \delta_\ell.$$

(10.53)

Therefore, we can write

$$\mathcal{I} \left[ f(k, 0) \right] = \frac{1}{k} \sum_{\ell} (2\ell + 1) \sin^2 \delta_\ell.$$

(10.54)
10.6. THE PHASE SHIFTS FOR TWO-BODY SCATTERING

Making use of Eq. (10.48), we can write

$$\Im m[f(k,0)] = \frac{k}{4\pi} \sigma_T . \quad (10.55)$$

This result is commonly known as the Optical Theorem, and relates the forward scattering amplitude to the total cross section. Considering the fact that the total cross section is proportional to the scattering amplitude squared, then Eq. (10.55) imposes a non-linear constraint on the scattering amplitude, \( f(k, \theta) \).

To illustrate the relation between unitarity and the optical theorem, we recall from Eq. (10.46) that the partial wave scattering amplitude is given by

$$f_\ell(k) = \frac{1}{2i} \left( e^{2i\delta_\ell} - 1 \right) = \frac{1}{2i} (S\ell(k) - 1) , \quad (10.56)$$

where \( S\ell(k) \) is the partial wave \( S \)-matrix element. Solving Eq. (10.52) for \( S\ell(k) \), we get

$$S\ell(k) = 1 + 2if_\ell(k) . \quad (10.57)$$

Unitarity is the result of the fact that the \( S \)-matrix is unitary, i.e.,

$$S^\dagger S = I , \quad (10.58)$$

which is obviously the case for \( S\ell(k) \) if the the phase shifts \( \delta_\ell \) are real. Also, the unitarity of \( S\ell(k) \) gives us the result

$$S^\dagger_\ell(k) S\ell(k) = 1 , \quad (10.59)$$

or

$$\Im m f_\ell(k) = |f_\ell(k)|^2 . \quad (10.60)$$

This non-linear relation for the partial wave scattering amplitude is identical to the optical theorem. We will see in a later chapter on formal scattering theory, Chapter 14, that this result can be derived from the Schrödinger equation directly.

10.6 The Phase Shifts for Two-Body Scattering

So far we have determined the cross section in terms of the scattering amplitude or the phase shifts. However, the phase shifts are constants used in writing the asymptotic form of the wave function. In the case of a square well potential these phase shifts were determined by matching the logarithmic derivative of the radial wave function at the well radius \( r = a \). In general, we can follow the same procedure and integrate the differential equation from the origin to a large enough radial distance \( r \), which is larger than the
CHAPTER 10. SCATTERING BY A CENTRAL POTENTIAL

range of the potential, and then match the logarithmic derivative, as calculated from the asymptotic solution and the solution we get by integrating the radial Schrödinger equation. This is achieved by taking \( r_0 \) to be such that \( r_0 \gg a \), where \( a \) is the range of the potential, and solving the radial Schrödinger equation for \( \psi_\ell(r) \) for \( r < r_0 \) and calculating the logarithmic derivative of \( \psi_\ell(r) \), i.e.,

\[
\gamma = \frac{d}{dr} \log \psi_\ell(r) \bigg|_{r = r_0 - \epsilon},
\]

where \( \epsilon \) is infinitesimal. On the other hand, for \( r > r_0 \), we have the asymptotic solution

\[
\psi_\ell(r) = \cos \delta_\ell \left( j_\ell(kr) + \tan \delta_\ell n_\ell(kr) \right)
\]

and its derivative

\[
\frac{d\psi_\ell}{dr} = k \cos \delta_\ell \left( j_\ell'(kr) + \tan \delta_\ell n_\ell'(kr) \right)
\]

where \( j_\ell' \) and \( n_\ell' \) are the derivative of the spherical Bessel and Neumann functions. We now can write the logarithmic derivative for \( r > r_0 \) as

\[
\frac{d}{dr} \log \psi_\ell(r) \bigg|_{r = r_0 + \epsilon} = k \frac{j_\ell'(kr_0) + \tan \delta_\ell n_\ell'(kr_0)}{j_\ell(kr_0) + \tan \delta_\ell n_\ell(kr_0)} = \gamma .
\]

This equation can be solved for the phase shift, or \( \tan \delta_\ell \), to give

\[
\tan \delta_\ell = \frac{k j_\ell'(kr_0) - \gamma j_\ell(kr_0)}{\gamma n_\ell(kr_0) - k n_\ell'(kr_0)} .
\]

Here, we observe that given \( \gamma \), we can determine the phase shift \( \delta_\ell \).

**Note:** Here \( \gamma \) is a function of \( r_0 \), and \( r_0 \) should be chosen large enough so that \( \delta_\ell \) or \( \tan \delta_\ell \) is independent of \( r_0 \).

From the above results we can study the behavior of \( \delta_\ell \) at low energies, i.e. \( k \to 0 \). For \( kr_0 \ll \ell \), we can write the spherical Bessel and Neumann functions and their derivative as

\[
\begin{align*}
 j_\ell(kr_0) &\to \frac{(kr_0)^\ell}{(2\ell + 1)!!} & &\text{and} & & n_\ell(kr_0) &\to \frac{(2\ell - 1)!!}{(kr_0)^{\ell+1}} \\
n_\ell'(kr_0) &\to -\frac{\ell (kr_0)^{\ell-1}}{(2\ell + 1)!!} & &\text{and} & & n_\ell'(kr_0) &\to \frac{(\ell + 1)(2\ell - 1)!!}{(kr_0)^{\ell+2}} .
\end{align*}
\]

We then can write the phase shift for \( kr_0 \ll \ell \) as

\[
\tan \delta_\ell \to \frac{(kr_0)^{2\ell+1}}{(2\ell + 1)!!(2\ell - 1)!!} \frac{\ell - \gamma r_0}{\ell + 1 + \gamma r_0} .
\]
or
\[ \tan \delta_\ell \propto k^{2\ell+1} \quad \text{for} \quad kr_0 \ll \ell . \] (10.66)

From this result we may deduce that for small wave number \( k \), i.e. low energy, \( \sin \delta_\ell \propto k^{2\ell+1} \). If we now write the cross section as
\[ \sigma_T = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \sin^2 \delta_\ell \equiv \sum_\ell \sigma_\ell , \] (10.67)

then the partial wave cross section, \( \sigma_\ell \), is given by
\[ \sigma_\ell = \frac{4\pi}{k^2} (2\ell + 1) \sin^2 \delta_\ell \]
\[ \rightarrow (\text{const.}) k^{4\ell} \quad \text{for} \quad kr_0 \ll \ell . \] (10.68)

From this result we may conclude that for \( k \to 0 \) we have
\[ \sigma_\ell = \begin{cases} \text{const.} & \text{for} \quad \ell = 0 \\ 0 & \text{for} \quad \ell \neq 0 \end{cases} . \] (10.69)

Thus at low energies, we expect the \( \ell = 0 \) partial wave to dominate the cross section.

### 10.7 Coulomb Scattering

The analysis in this chapter has so far been restricted to finite range potentials. This excludes the Coulomb potential which is considered to be infinite in range. Because of the central role played by the Coulomb potential in both atomic and molecular physics, and the fact that all accelerators produce beams of charged particles which are scattered by targets often made of charged particles (e.g. nuclei), a discussion of scattering theory that excludes the Coulomb problem is incomplete. The aim of this section is to derive the amplitude for the scattering of two charged particles, and from that, extract the Rutherford cross section.

Consider for the present, the scattering of two charged particles with charges \( Z e \) and \( Z' e' \), with the Coulomb potential between the particles given by
\[ V(\vec{r}) = \frac{ZZ'ee'}{r} . \] (10.70)

The Schrödinger equation for this potential is then given by
\[ \left( -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{ZZ'ee'}{r} \right) \psi(\vec{r}) = E \psi(\vec{r}) , \] (10.71)
where $\mu$ is the reduced mass of the two particles. We can rewrite this equation, after multiplication by $-\frac{2\mu}{\hbar^2}$, as

$$\left(\nabla^2 + k^2 - \frac{2\gamma k}{r}\right)\psi(\vec{r}) = 0,$$

(10.72)

where

$$k^2 = \frac{2\mu E}{\hbar^2} \quad \text{and} \quad \gamma = \frac{\mu Z Z' e e'}{\hbar^2 k}. \quad (10.73)$$

We are going to consider the solution of this equation for $E > 0$, i.e., we would like to derive the scattering wave function $\psi(\vec{r})$, and from its asymptotic behavior extract the scattering amplitude.

For any scattering experiment, we can define the direction of the incident beam to be the $z$-axis. The presence of this preferred direction in space breaks the symmetry (that space is isotropic) and it gives us a problem which has cylindrical symmetry. As a result of this new symmetry we expect the wave function and the scattering amplitude to be independent of the angle $\phi$, as illustrated in Eq. (10.32). Furthermore, the wave function asymptotically should have an incident beam given up to a normalization by

$$e^{ikz},$$

and a scattered beam that is proportional to a spherical out going wave, i.e.,

$$\frac{e^{ikr}}{r}.$$  

This suggests that the solution to Eq. (10.72) can be written as

$$\psi(\vec{r}) = e^{ikz} g(r - z),$$

and excludes the possibility of having a function of the form

$$\psi(\vec{r}) = e^{ikz} g(r + z),$$

since the latter leads to an incoming spherical wave given by

$$\frac{e^{-ikr}}{r}.$$  

The fact that the boundary condition on this scattering problem requires the total wave function to be a function of $z$ and $r - z$, suggests that the optimum choice for a coordinate system for solving the Schrödinger equation is the parabolic coordinate given by

$$\begin{align*}
\xi &= r - z = r(1 - \cos \theta) \\
\eta &= r + z = r(1 + \cos \theta) \\
\phi &= \phi.
\end{align*}$$

(10.74)
In this new coordinate system, our total scattering wave function is a product of an incident plane wave and a function of $\xi$, i.e.,

$$\psi(\vec{r}) = e^{i(k_\eta - \xi)/2} g(\xi). \quad (10.75)$$

The Laplacian, $\nabla^2$, in this coordinate system is given by

$$\nabla^2 = \left[ \frac{1}{\xi + \eta} \left\{ \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \eta} \right) \right\} + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \phi^2}. \right] \quad (10.76)$$

We now can write the Schrödinger equation in this coordinate system as a partial differential equation in two variables, since we have no $\phi$ dependence. Furthermore, for the wave function with the structure given in Eq. (10.75), we have

$$\nabla^2 \psi(\vec{r}) = \frac{4}{\eta + \xi} \left[ \frac{\xi d^2 g}{d\xi^2} + (1 - ik \xi) \frac{dg}{d\xi} - \frac{k^2}{4} (\eta + \xi) g \right]. \quad (10.77)$$

This allows us to rewrite Eq. (10.72) for $g(\xi)$ as

$$\xi \frac{d^2 g}{d\xi^2} + (1 - ik \xi) \frac{dg}{d\xi} - \gamma kg(\xi) = 0, \quad (10.78)$$

which is the Confluent Hypergeometric equation, with the solution given by

$$g(\xi) = F(-i\gamma |1| ik \xi), \quad (10.79)$$

where the Confluent Hypergeometric function $F(\alpha |\beta | z)$ is given in Eq. (8.35) as an infinite series. We thus can write the total wave function for the scattering of two charged particles as

$$\psi(\vec{r}) = e^{ikz} g(r - z) = e^{ikz} F(-i\gamma |1| ik(r - z)). \quad (10.80)$$

To get the asymptotic form of this wave function, and thus determine the scattering amplitude, we need to know the behavior of $F(\alpha |\beta | \rho)$ for large $\rho$. We have that\footnote{See Abramowitz and Stegun [24] Eq. 13.5.1}

$$F(\alpha |\beta | \rho) \to \Gamma(\beta) \left\{ \frac{(-\rho)^{-\alpha}}{\Gamma(\beta - \alpha)} + \frac{\rho^{\alpha-\beta} e^\rho}{\Gamma(\alpha)} \right\} \text{ as } |\rho| \to \infty. \quad (10.81)$$

With this result for the asymptotic behavior of the Confluent Hypergeometric equation, we can write the total scattering wave function for $r \to \infty$ as

$$\psi(\vec{r}) \to e^{\pi\gamma/2} \Gamma(1 + i\gamma) \left\{ e^{i(kz + \gamma \log k(r - z))} + \frac{\Gamma(1 + i\gamma) e^{-i\gamma \log \sin^2 \theta/2}}{2k^2 \sin^2 \theta/2} e^{i(kr - \gamma \log 2kr)} \right\} \to e^{\pi\gamma/2} \Gamma(1 + i\gamma) \left[ e^{i(kz + \gamma \log k(r - z))} + f(k, \theta) \frac{e^{i(kr - \gamma \log 2kr)}}{r} \right], \quad (10.82)$$
where the scattering amplitude \( f(k, \theta) \) is given by \(^7\)

\[
f(k, \theta) = -\frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)} \frac{e^{-i\gamma \log \sin^2 \theta/2}}{2k \sin^2 \theta/2}
= -\frac{\gamma}{2k \sin^2 \theta/2} e^{i(\eta_0 - \gamma \log \sin^2 \theta/2)}, \tag{10.83}
\]

where

\[
\eta_0 = \arg \Gamma(1 + i\gamma). \tag{10.84}
\]

The corresponding differential cross section is then given by

\[
\frac{d\sigma}{d\Omega} = \frac{\gamma^2}{4k^2 \sin^4 \theta/2}. \tag{10.85}
\]

This cross section is commonly known as the Rutherford cross section. Here we observe that:

1. The differential cross section for the scattering of two charged particles, \( \frac{d\sigma}{d\Omega} \to \infty \) as \( \theta \to 0 \). This in fact is the case both for electron scattering on atoms and proton scattering off a nucleus.

2. The total cross section is also infinity. This is a result of the fact that the Coulomb potential has an infinite range.

3. The asymptotic wave function as presented in Eq. (10.82) consists of an incident plane wave and an outgoing spherical scattering wave. However, if we compare this result with the equation we got for finite range potentials, i.e., Eq. (10.42), we observe that both the plane wave and the scattering wave are modified by a factor proportional to \( \gamma \), which is basically the product of the charges on the two particles. This distortion of the incident plane wave and the scattered spherical wave are the result of the infinite range of the Coulomb potential.

### 10.8 Problems

1. Calculate the S-wave phase shift (i.e. \( \ell = 0 \)) for neutron proton scattering at center of mass energies of 5 and 10 MeV given that the potential between a neutron and

\(^7\)In writing this result we have made use of the fact that

\[
(1 - \cos \theta) = 2 \sin^2 \theta/2,
\]

and

\[
-z \Gamma(-z) = \Gamma(1-z).
\]
proton is a square well of radius \( r_0 = 2.51 \text{ fm} \) and the depth is \( V_0 = 17.8 \text{ MeV} \). Calculate the corresponding cross section.

**Note:** 1 fm. = \( 10^{-13} \text{ cm} \), and \( \hbar^2/2\mu = 41.47 \text{ MeV fm}^2 \).

2. Calculate the cross section for scattering off a hard sphere of radius \( R \) at very low energies, i.e. in the limit where \( kR \ll 1 \).

**Hint:** A hard sphere can be represented by the potential
\[
V(r) = \begin{cases} 
+\infty & r \leq R \\
0 & r > R 
\end{cases}
\]

3. Show that for \( P \)-wave (\( \ell = 1 \)) scattering by an attractive square well potential of radius \( a \) and depth \( V_0 \), the phase shifts satisfy the equation
\[
\alpha^2 [ka \cot (ka + \delta_1) - 1] = k^2 (\alpha a \cot \alpha a - 1),
\]
where \( k^2 = \frac{2m}{\hbar^2} E \) and \( \alpha^2 = \frac{2m}{\hbar^2} (E + V_0) \). For neutron proton scattering we can take \( V_0 = 36.2 \text{ MeV} \) and \( a = 2.02 \text{ fm} \). Taking \( \frac{\hbar^2}{2\mu} = 41.47 \text{ MeV fm}^2 \), calculate the phase shift at \( E = 10 \text{ MeV} \).

4. Using the orthogonality of the Legendre polynomials, show that if the scattering wave function is written as
\[
\psi(r) = \frac{1}{(2\pi)^{3/2}} \sum_\ell i^\ell (2\ell + 1) \psi_\ell(r) P_\ell(\cos \theta),
\]
then \( \psi_\ell(r) \) satisfies the radial Schrödinger equation.

5. Show that for complex phase shifts the total cross section as calculated from the Optical Theorem is larger than the total elastic cross section obtained from integrating the differential cross section.

**Hint:** To prove the above you can restrict your argument to one partial wave, e.g. \( \ell = 0 \).

6. The amplitude of \( S \)-wave scattering at at low energies is given by
\[
f_0 = -\frac{ak}{1 + iak - \frac{1}{2}ar_e k^2}
\]
where \( a \) and \( r_e \) are real constants, and \( k \) is the momentum.

(a) Show that this amplitude satisfies unitarity.

(b) Show that the corresponding cross section goes to a constant as the energy goes to zero, i.e. \( k \to 0 \).
Chapter 14

Scattering Theory; Revisited

In Chapter 2 we showed how we can calculate the cross section for two particle scattering in terms of the scattering amplitude $f(k, \theta)$, and how this scattering amplitude is related to the phase shifts $\delta_l$. In particular, we found that we needed to solve the Schrödinger equation for the wave function $\psi_l(r)$ for all $r$ and then extract the phase shifts, knowing that for $r \to \infty$

$$\psi_l(r) \to \frac{1}{kr} \sin(kr - \frac{1}{2}\pi l + \delta_l) .$$

In other words, the asymptotic wave function determines the scattering amplitude and, therefore cross section, yet we need to calculate the wave function for all $r$. Since the wave function is not the observable we measure, we would like to set up an equation for the scattering amplitude $f(k, \theta)$ which is the observable.

In this chapter we will first derive an equation, the Lippmann-Schwinger equation, which can be solved for the scattering amplitude. We will find that this equation allows us not only to derive the Optical Theorem, but also to consider perturbation expansion for the scattering amplitude in the form of the Born series. Finally, we will consider a special class of potentials, called separable potentials, where we can study the properties of the scattering amplitude analytically.

14.1 Formal Theory of Scattering

To derive an equation for the scattering amplitude $f(k, \theta)$, we need to convert the Schrödinger equation plus boundary condition into an integral equation which incorporates these boundary conditions. Consider the Hamiltonian

$$H = H_0 + V ,$$

(14.1)

where $H_0$ is taken to be the kinetic energy in the center of mass, i.e., $H_0 = \frac{p^2}{2\mu}$ with $\mu$ the reduced mass of the system given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} .$$
Here \( m_1 \) and \( m_2 \) are the masses of the two particles in the collision. In Eq. (14.1), \( V \) is the interaction between the two particles. We also introduce the eigenstates of \( H_0 \) to be

\[
H_0 |\phi_\vec{k}\rangle = E |\phi_\vec{k}\rangle ,
\]

where \( E = \frac{\hbar^2 k^2}{2\mu} \). These states are labeled by the initial momentum \( \vec{p} = \hbar \vec{k} \). In coordinate representation, the state \( |\phi_\vec{k}\rangle \) is a plane wave, i.e.,

\[
\langle \vec{r} | \phi_\vec{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i \vec{k} \cdot \vec{r}}.
\]

We note here that this state is identical to the eigenstate of the momentum operator as given in Eq. (11.93), i.e. \( \langle \vec{r} | \vec{k} \rangle = \langle \vec{r} | \phi_\vec{k} \rangle \). The Schrödinger equation for the full Hamiltonian \( H \) can now be written as

\[
(E - H_0) |\psi\rangle = V |\psi\rangle .
\]

This can be thought of as an inhomogeneous differential equation if we take it in coordinate representation, and in this case, the solution can be written as the sum of a particular solution plus a solution to the homogeneous equation. The particular solution is given by

\[
|\psi\rangle = \frac{1}{E - H_0} V |\psi\rangle .
\]

In writing the operator \((E - H_0)^{-1}\), we have assumed that the operator \((E - H_0)\) has no zero eigenvalues. In fact, as we will find, the operator \((E - H_0)\) does have zero eigenvalues and therefore its inverse which appears in Eq. (14.5) is singular. We will show next how the boundary conditions in the Schrödinger equation are used to overcome these singularities. The general solution of Eq. (14.4) is then given by

\[
|\psi_\vec{k}\rangle = |\phi_\vec{k}\rangle + \frac{1}{E - H_0} V |\psi_\vec{k}\rangle .
\]

We have labeled our solution \( |\psi_\vec{k}\rangle \) by the vector \( \vec{k} \) to indicate that the momentum of the incident beam is \( \hbar \vec{k} \). This in turn means that the energy of the incident beam in the center of mass is given by \( E = \frac{\hbar^2 k^2}{2\mu} \).

To examine the singularities of \((E - H_0)^{-1}\), we write Eq. (14.6) in coordinate representation as

\[
\langle \vec{r} | \psi_\vec{k} \rangle = \langle \vec{r} | \phi_\vec{k} \rangle + \int d^3r \langle \vec{r} |(E - H_0)^{-1}|\vec{r}'\rangle \langle \vec{r}' | V |\psi_\vec{k}\rangle .
\]

We are now in a position to examine the singularities of the Green’s function.\footnote{This Green’s function is identical to the Green’s function encountered in the theory of differential equations. After all, we are solving the Schrödinger equation which is a second order differential equation.} Making use of the fact that the complete set of eigenstates of the momentum operator are also...
14.1. FORMAL THEORY OF SCATTERING

Since the eigenvalue of $H_0$ is $\frac{\hbar^2 k^2}{2\mu}$, taking the energy $E$, which is the energy of the initial incident particle in the two-body center mass, to be related to the incident momentum, $\hbar k_0$, then the integral in Eq. (14.8) can be written as

$$
\langle \vec{r}' | (E - H_0)^{-1} | \vec{r}'' \rangle = \frac{1}{(2\pi)^3} \frac{2\mu}{\hbar^2} \int d^3k \frac{e^{i\vec{k} \cdot (\vec{r}' - \vec{r}'')}}{k_0^2 - k^2} 
$$

The integrand now has two poles at $k = \pm k_0$, which is on the integration path. To overcome this problem, we need to go around these poles. This can be achieved by taking

$$
k_0^2 \to k_0^2 \pm i\epsilon ,
$$

where $\epsilon$ is an infinitesimal quantity. We then perform the integral and take the limit of $\epsilon \to 0$. As we will see, the choice of sign in $k_0^2 \pm i\epsilon$ will determine the boundary condition. Let us take the positive sign, i.e., $k_0^2 + i\epsilon$, for the present. Then our Green’s function has the integral representation given by

$$
\langle \vec{r}' | (E - H_0)^{-1} | \vec{r}'' \rangle = \frac{1}{i(2\pi)^2} \frac{2\mu}{\hbar^2} \left| \vec{r}' - \vec{r}'' \right| \int_{-\infty}^{+\infty} dk \frac{k^2}{k_0^2} \frac{e^{ik\left| \vec{r}' - \vec{r}'' \right|}}{k_0^2 + i\epsilon - k^2} ,
$$

The integrand now has two poles at

$$
k = \pm (k_0 + i\epsilon) .
$$

---

\[2\]This is one choice of integration path. Other choices will entail taking the integration path below or above both poles. These correspond to other boundary conditions.
We now can perform the integral in Eq. (14.11) by making use of Cauchy’s Theorem. For \( k \) in the upper half of the complex \( k \)-plane, i.e., \( \text{Im}(k) > 0 \), the integrand has the property that it goes to zero as \( |k| \to \infty \). This condition allows us to convert the integral over the interval \( -\infty \) to \( +\infty \) to an integral along a contour \( C \) in Figure 14.1 where the semi-circle corresponds to \( |k| \to \infty \). Because the integrand is zero on the infinite semi-circle it does not contribute to the value of the Green’s function. This allows us to write the Green’s function as

\[
\langle \vec{r}' | (E - H_0)^{-1} | \vec{r}'' \rangle \quad = \quad \frac{1}{i(2\pi)^2} \frac{2\mu}{\hbar^2} \oint_C dk \frac{e^{ik|\vec{r}' - \vec{r}''|}}{(k_0 + i\epsilon - k)(k_0 + i\epsilon + k)}
\]

\[
= -\frac{1}{4\pi^2} \frac{2\mu}{\hbar^2} e^{ik_0|\vec{r}' - \vec{r}''|},
\]

where we have made use of Cauchy’s Residue Theorem\(^3\) and have taken the limit \( \epsilon \to 0 \).

Had we chosen \( k_0^2 \to k_0^2 - i\epsilon \), the poles in the integrand would have been at

\[
k = \pm(k_0 - i\epsilon).
\]

In this case the contour \( C \) would have enclosed the pole at \( k = -k_0 + i\epsilon \), and we would get the factor of

\[
e^{-ik_0|\vec{r}' - \vec{r}''|}
\]

---

\(^3\)Cauchy’s theorem allows us to write the integral over a closed contour in terms of the sum over the residues of all the poles in the integrand that are inside the contour, i.e.,

\[
\oint_C dk f(k) = 2\pi i \sum_n \text{Res}[f(k)],
\]

where \( n \) runs over all the poles inside the contour \( C \).
in Eq. (14.12) instead of
\[ e^{i k_0 |\mathbf{r} - \mathbf{r}'|}. \]
This we will see corresponds to spherical waves which are moving towards the scattering center, while the Green’s function in Eq. (14.12) corresponds to spherical outgoing waves. These two Green’s functions can now be written as
\[ \langle \mathbf{r}' | (E^\pm - H_0)^{-1} | \mathbf{r}'' \rangle = \lim_{\epsilon \to 0} \langle \mathbf{r}' | (E \pm i \epsilon - H_0)^{-1} | \mathbf{r}'' \rangle \]
\[ = \mp \frac{1}{4\pi} \frac{2\mu}{\hbar^2} \frac{e^{\pm ik_0 |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}. \]  \hfill (14.13)
With this result in hand we can write the scattering wave function, in coordinate representation, by substituting the Green’s function \( \langle \mathbf{r}' | (E^+ - H_0)^{-1} | \mathbf{r}'' \rangle \) in Eq. (14.7) to get
\[ \langle \mathbf{r}' | \psi_{\mathbf{k}_0}^{(+)} \rangle = \langle \mathbf{r}' | \phi_{\mathbf{k}_0} \rangle - \frac{1}{4\pi} \frac{2\mu}{\hbar^2} \int d^3 \mathbf{r}' e^{i k_0 |\mathbf{r} - \mathbf{r}'|} \langle \mathbf{r}' | V | \psi_{\mathbf{k}_0}^{(+)} \rangle, \]  \hfill (14.14)
where \((+)\) superscript on the wave function indicates that the boundary condition corresponds to an outgoing spherical wave. To extract the scattering amplitude out of this wave function we need to take the limit as \( r \to \infty \) to determine the coefficient of the outgoing spherical wave and compare that equation with Eq. (10.3) to determine the scattering amplitude. To take the limit as \( r \to \infty \) we make use of the fact that
\[ |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2 \mathbf{r} \cdot \mathbf{r}'} \]
\[ = r \left\{ 1 + \left( \frac{r'}{r} \right)^2 - 2 \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} \right\}^{1/2} \]
\[ \to r - \hat{\mathbf{r}} \cdot \mathbf{r}' \quad \text{for} \quad r \to \infty, \]
where \( \hat{\mathbf{r}} \) is a unit vector in the \( \mathbf{r}' \) direction. We now can write the asymptotic wave function (i.e., \( r \to \infty \)) as
\[ \langle \mathbf{r}' | \psi_{\mathbf{k}_0}^{(+)} \rangle \to \frac{1}{(2\pi)^{3/2}} e^{ik_0 \cdot \mathbf{r}} - \frac{1}{4\pi} \frac{2\mu}{\hbar^2} \int d^3 \mathbf{r}' e^{-i k_0 \hat{\mathbf{r}} \cdot \mathbf{r}'} \langle \mathbf{r}' | V | \psi_{\mathbf{k}_0}^{(+)} \rangle. \]  \hfill (14.15)
Since the unit vector \( \hat{\mathbf{r}} \) is in the same direction as the final momentum of the scattered particle, and the magnitude of the initial and final momentum are the same due to energy conservation, we can take the final momentum to be \( \mathbf{k}_f = k_0 \hat{\mathbf{r}} \). Making use of this result and the fact that
\[ \langle \mathbf{k}_f | \phi_{\mathbf{k}_f} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i k_f \cdot \mathbf{r}'}, \]
we can write the asymptotic wave function as
\[ \langle \mathbf{r}' | \psi_{\mathbf{k}_i}^{(+)} \rangle \to \frac{1}{(2\pi)^{3/2}} \left\{ e^{ik_i \cdot \mathbf{r}} - \frac{4\pi^2 \mu}{\hbar^2} \langle \mathbf{k}_f | V | \psi_{\mathbf{k}_i}^{(+)} \rangle \frac{e^{ik_f}}{r} \right\}. \]  \hfill (14.16)
In writing the above scattering wave function, we have labeled the initial momentum, i.e., the momentum of the incident beam, by \( \vec{k}_i \), while the final momentum is labeled as \( \vec{k}_f \). Conservation of energy then requires that the magnitude of the initial and final momentum be equal if the scattering is elastic, i.e., \(|\vec{k}_i| = |\vec{k}_f| \equiv k\). Comparing Eqs. (14.16) and (10.3) allows us to write the scattering amplitude in terms of the matrix elements of the potential as

\[
f(k, \theta) = -\frac{4\pi^2 \mu}{\hbar^2} \langle \phi_{\vec{k}_i} | V | \psi_{\vec{k}_f}^{(+)} \rangle ,
\]

(14.17)

where \( \cos \theta = \vec{k}_f \cdot \vec{k}_i \). Since the scattering amplitude is basically an observable, we will introduce a corresponding operator which we will call the \( T \)-matrix, and which is defined by the relation

\[
V | \psi_{\vec{k}_i}^{(+)} \rangle \equiv T(E^+) | \phi_{\vec{k}_i} \rangle ,
\]

(14.18)

where the boundary condition is now specified by labeling the \( T \)-matrix with \( E^+ = E + i\epsilon \).

We now can write the scattering amplitude in terms of the \( T \)-matrix as

\[
f(k, \theta) = -\frac{4\pi^2 \mu}{\hbar^2} \langle \phi_{\vec{k}_i} | T(E^+) | \phi_{\vec{k}_i} \rangle \tag{14.19a}
\]

\[
= -\frac{4\pi^2 \mu}{\hbar^2} \langle \vec{k}_f | T(E^+) | \vec{k}_i \rangle . \tag{14.19b}
\]

Therefore, the problem of finding an equation for the scattering amplitude has been reduced to the problem of finding an equation for the \( T \)-matrix. Making use of the definition of the \( T \)-matrix as given in Eq. (14.18), we can write the scattering wave function given in Eq. (14.6) as

\[
| \psi_{\vec{k}_i}^{(+)} \rangle = | \phi_{\vec{k}_i} \rangle + G_0(E^+) T(E^+) | \phi_{\vec{k}_i} \rangle ,
\]

(14.20)

where \( G_0(E^+) = (E^+ - H_0)^{-1} \) is the Green’s function in operator form. This Green’s function is often referred to as the free-particle Green’s function since it is the Green’s function for the Schrödinger equation in the absence of any interaction. We now multiply this equation from the left by the potential \( V \), to get

\[
V | \psi_{\vec{k}_i}^{(+)} \rangle = V | \phi_{\vec{k}_i} \rangle + V G_0(E^+) T(E^+) | \phi_{\vec{k}_i} \rangle . \tag{14.21}
\]

Applying the definition of the \( T \)-matrix, Eq. (14.18), to the left hand side of Eq. (14.21) gives us the equation

\[
T(E^+) | \phi_{\vec{k}_i} \rangle = V | \phi_{\vec{k}_i} \rangle + V G_0(E^+) T(E^+) | \phi_{\vec{k}_i} \rangle . \tag{14.22}
\]

Since this is valid for any state \( | \phi_{\vec{k}_i} \rangle \), we can write an operator equation for the \( T \)-matrix which is of the form

\[
T(E^+) = V + V G_0(E^+) T(E^+) . \tag{14.23}
\]
We have taken the operator $T$ to be a function of the energy $E^+$ since the Green’s function $G_0(E^+)$ is a function of the energy, and we need to know the Green’s function to determine the $T$-matrix. Also, by specifying the energy in the form of $E^+$, we have specified the boundary condition that we have spherically outgoing waves. Equation (14.23) is known as the Lippmann-Schwinger equation, and is equivalent to the Schrödinger equation including the boundary conditions. The solution of this equation will give us the scattering amplitude directly and thus the cross section.

To solve Eq. (14.23), we need to write the equation in a given representation. In this case the natural representation is the momentum representation which gives us

$$
\langle \vec{k} | T(E^+) | \vec{k}' \rangle
$$

and then the scattering amplitude, as measured in elastic scattering, is given by

$$
\langle \vec{k} | T(E^+) | \vec{k}' \rangle \quad \text{with} \quad |\vec{k}| = |\vec{k}'| \quad \text{and} \quad E = \frac{\hbar^2 k^2}{2\mu} .
$$

(14.24)

The Lippmann-Schwinger equation, Eq. (14.23), can be written in momentum space as

$$
\langle \vec{k} | T(E^+) | \vec{k}' \rangle = \langle \vec{k} | V | \vec{k}' \rangle + \langle \vec{k} | G_0(E^+) T(E^+) | \vec{k}' \rangle
$$

$$
= \langle \vec{k} | V | \vec{k}' \rangle + \int d^3k'' \frac{\langle \vec{k} | V | \vec{k}'' \rangle \langle \vec{k}'' | T(E^+) | \vec{k}' \rangle}{E + i\epsilon - \frac{\hbar^2 k''^2}{2\mu}} .
$$

(14.25)

In writing the second line of the above equation we have made use of the fact that the eigenstates of the momentum operator are complete, and that these states are also eigenstates of $H_0$ with eigenvalue $\frac{\hbar^2 k^2}{2\mu}$. In Eq. (14.25), we have an integral equation in three-dimensions which is very difficult to solve as it stands. To reduce the dimensionality of the equation, we need to partial wave expand the matrix elements of the potential $V$ and the $T$-matrix $T(E^+)$. This expansion involves the introduction of eigenstates of the total angular momentum and its projection along the $z$-axis. In the case of the scattering of spin-less particles, the total angular momentum is the orbital angular momentum. For the potential $V$, the partial wave expansion involves writing the matrix elements of $V$ as

$$
\langle \vec{k} | V | \vec{k}' \rangle = \langle \vec{k}; k | V | k'; \vec{k}' \rangle
$$

$$
= \sum_{\ell m} \langle \vec{k} | \ell m \rangle \langle \ell m, k | V | k', \ell m \rangle \langle \ell m | \vec{k}' \rangle
$$

(14.26)

Note that the eigenstates of the momentum operator and the free Hamiltonian $H_0$ are identical with our choice of normalization, i.e.,

$$
|\phi_\vec{k}\rangle = |\vec{k}\rangle .
$$

In writing Eq. (14.26), we are making use of the notation

$$
|\vec{k}\rangle = |k; \vec{k}\rangle = |k\rangle |\vec{k}\rangle
$$

to separate the radial from the angular variables.
\[ \equiv \sum_{\ell m} \langle \hat{k}|\ell m \rangle \langle k|V|k' \rangle \langle \ell m|\hat{k}' \rangle . \] (14.26)

In writing the first line of the above equation we have separated the radial from the angular part of the momentum eigenstates. We have also taken into consideration that the potential is spherically symmetric by taking the matrix elements of \( V \) to be diagonal in \( \ell \) and independent of \( m \), i.e., since \([H, L^2] = [H, L_z] = 0\), we have

\[ \langle \ell m, k|V|k', \ell' m' \rangle = \delta_{\ell \ell'} \delta_{mm'} \langle k|V|k' \rangle . \] (14.27)

The independence of the matrix elements from \( m \) is a result of the fact that the radial wave functions for a spherically symmetric potential are independent of \( m \). In Eq. (14.26) the bracket \( \langle \hat{k}|\ell m \rangle \) is nothing other than the spherical harmonic, i.e.,

\[ \langle \hat{k}|\ell m \rangle = Y_{\ell m}(\hat{k}) . \]

In other words the expansion in Eq. (14.26) is similar to that in Eq. (10.45) if we take into consideration that the Legendre polynomial can be written in terms of spherical harmonics using the addition theorem, i.e.,

\[ \sum_m Y_{\ell m}(\hat{k}) Y_{\ell m}^*(\hat{k}') = \frac{2\ell + 1}{4\pi} P_\ell(\cos \theta) . \] (14.28)

In this case we can write the potential in momentum representation as

\[ \langle \hat{k}|V|\hat{k}' \rangle = \frac{1}{4\pi} \sum_\ell (2\ell + 1) V_\ell(k, k') P_\ell(\cos \theta) , \] (14.29)

with

\[ V_\ell(k, k') = \langle k|V|k' \rangle , \] (14.30)

and \( \cos \theta = \hat{k} \cdot \hat{k}' \). In a similar manner we can write a partial wave expansion for the \( T \)-matrix which is of the form

\[ \langle \hat{k}|T(E^+)|\hat{k}' \rangle = \sum_{\ell m} \langle \hat{k}|\ell m \rangle \langle k|T_\ell(E^+)|k' \rangle \langle \ell m|\hat{k}' \rangle \]

\[ = \frac{1}{4\pi} \sum_\ell (2\ell + 1) T_\ell(k, k'; E^+) P_\ell(\cos \theta) . \] (14.31)

With these partial wave expansions for the potential \( V \) and the \( T \)-matrix \( T(E^+) \), and using the orthogonality of the spherical harmonics, i.e.,

\[ \int d\hat{k} \langle \ell m|\hat{k} \rangle \langle k|\ell' m' \rangle = \delta_{\ell \ell'} \delta_{mm'} , \] (14.32)
we can write the Lippmann-Schwinger equation as a one dimensional integral equation of the form
\[ \langle k | T_\ell(E^+) | k' \rangle = \langle k | V_\ell | k' \rangle + \int_0^\infty dk'' k''^2 \frac{\langle k | V_\ell | k'' \rangle \langle k'' | T_\ell(E^+) | k' \rangle}{E + i\epsilon - \frac{\hbar^2 k''^2}{2\mu}}, \] (14.33a)
or
\[ T_\ell(k, k'; E^+) = V_\ell(k, k') + \int_0^\infty dk'' \frac{k''^2 V_\ell(k, k'')}{E + i\epsilon - \frac{\hbar^2 k''^2}{2\mu}} T_\ell(k'', k'; E^+) . \] (14.33b)

This equation can be solved numerically on any present day computer. This is achieved by replacing the integral by a sum and in this way we turn the integral equation into a set of linear algebraic equations. In converting the above integral equation into a set of algebraic equations we should note that the energy \( E \) and the initial momentum \( k' \) are parameters and play no role in the solution of the equation. In fact, to get the physical amplitude we should take
\[ E = \frac{\hbar^2 k_0^2}{2\mu} \quad \text{and} \quad k' = k_0 \]
where \( k_0 \) is referred to as the *on-shell* momentum.

By comparing the partial wave expansion of the scattering amplitude, \( f(k, \theta) \), as given in Eq. (10.45) with the expansion of the \( T \)-matrix as given in Eq. (14.31), and making use of Eq. (14.19), we can write the partial wave scattering amplitude \( f_\ell(k) \) in terms of the partial wave \( T \)-matrix \( T_\ell(k_0, k_0; E^+) \) as
\[ f_\ell(k_0) = \frac{1}{2i} \left( e^{2i\delta_\ell} - 1 \right) = -\frac{\pi \mu k_0}{\hbar^2} T_\ell(k_0, k_0; E^+) \equiv -\frac{\pi \mu k_0}{\hbar^2} T_\ell(k_0) . \] (14.34)

To solve Eq. (14.33) for the scattering amplitude, we need to determine the input partial wave potential \( V_\ell(k, k') \), and the Green’s function \( G_0(E^+) \) in momentum representation. To determine the partial wave potential we need to first write the potential in momentum representation in terms of the potential in coordinate space as we have used it in previous chapters. All the potentials we have encountered to date are diagonal in coordinate representation, i.e.,
\[ \langle \vec{r} | V | \vec{r}' \rangle = \delta(\vec{r} - \vec{r}') V(\vec{r}) . \] (14.35)

These potentials are known as local potentials in contrast to non-local potentials that are not diagonal in coordinate representation. For local potentials, we have in momentum space
\[ \langle \vec{k} | V | \vec{k}' \rangle = \int d^3r \ d^3r' \langle \vec{k} | \vec{r} \rangle \langle \vec{r} | V | \vec{r}' \rangle \langle \vec{r}' | \vec{k}' \rangle = \int d^3r \langle \vec{k} | \vec{r} \rangle V(\vec{r}) \langle \vec{r} | \vec{k}' \rangle . \] (14.36)
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Making use of the partial wave expansion of the eigenstates of the momentum operator, i.e.,

\[
\langle \vec{r} | \hat{\vec{k}} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i \vec{k} \cdot \vec{r}}
\]

\[
= \sqrt{\frac{2}{\pi}} \sum_{\ell m} \frac{4}{\ell} j_{\ell}(kr) Y_{\ell m}(\hat{\vec{r}}) Y_{\ell m}^*(\hat{\vec{k}}) ,
\]

(14.37)

and the orthogonality of the spherical harmonics in Eq. (14.36) allows us to write the potential in momentum space as

\[
\langle \vec{k} | V | \vec{k}' \rangle = \sum_{\ell m} Y_{\ell m}(\hat{\vec{k}}) V_{\ell}(k, k') Y_{\ell m}^*(\hat{\vec{k}}') ,
\]

(14.38)

where

\[
V_{\ell}(k, k') = \frac{2}{\pi} \int_0^{\infty} dr \int_0^{\infty} dr' \frac{r^2}{r'^2} j_{\ell}(kr) V_{\ell}(r, r') j_{\ell}(k'r')
\]

(14.39a)

\[
= \frac{2}{\pi} \int_0^{\infty} dr \int_0^{\infty} dr' j_{\ell}(kr) V_{\ell}(r) j_{\ell}(k'r) .
\]

(14.39b)

Here, Eq. (14.39b) is to be used for the special case when the potential is local and central, in which case \( V_{\ell}(r) = V(r) \). Thus for any local central potential, Eq. (14.39b) defines the momentum space partial wave potential for use in the integral equation given in Eq. (14.33). The solution of this Lippmann-Schwinger equation then gives us the scattering amplitude for a given angular momentum \( \ell \). To get the differential cross section, we need to solve Eq. (14.33) for all partial waves that are important. For finite range potentials, the maximum value of \( \ell \) can be estimated by taking the classical argument that the maximum angular momentum is the cross product of the incident momentum times the maximum impact parameter, which in this case is the range of the potential, i.e., \( \ell_{\text{max}} = r_0 k \), where \( r_0 \) is the range of the potential.\(^6\)

14.2 The Born Approximation

The solution of the Lippmann-Schwinger equation, Eq. (14.33), is often not simple to get, and one may need to resort to approximation methods. One approximation often used at high energies or weak potentials in both atomic and nuclear physics is the Born approximation. This is the first term in a series similar to the perturbation series we

\(^6\)Here, we should note that for the case of the Coulomb potential, where the range of the interaction is infinite, the partial wave sum should be examined carefully.
developed in the last chapter. To get the series, we iterate Eq. (14.23) to get, in operator form, the Born series given by

\[ T(E^\pm) = V + V G_0(E^\pm) V + V G_0(E^\pm) G_0(E^\pm) V + \cdots \]  

(14.40)

This is a power series in the potential, and therefore for a potential that is weak, the series converges. Here, as in the case of perturbation theory for bound states, we can define \( H_0 \) to include some of the interaction and in this way, the remaining interaction \((H - H_0)\) is weak enough for the Born series to converge. However, in this case, to get the amplitude or \( T \)-matrix, we need to take the matrix element of \((H - H_0)\) with respect to the solution of the Schrödinger equation with the Hamiltonian \( H_0 \). Also the Green’s function \( G_0(E^\pm) \) has to be replaced by the Green’s function for the Hamiltonian \( H_0 \) which includes the interaction. This procedure has been implemented for Coulomb plus short range potential, where we know the solution for the Coulomb Hamiltonian analytically.

Another condition under which the above Born series converges is when the energy \( E \) is high.\(^7\) In that case the series is again a power series in \( E^{-1} \), and the first few terms of the series give a good approximation.

The Born approximation is used when the first term in the Born series is taken to be the amplitude for scattering. For this case, we have

\[ \langle \vec{k} | T(E^+) | \vec{k}' \rangle \approx \langle \vec{k} | V | \vec{k}' \rangle = \langle \vec{k} | T_B | \vec{k}' \rangle \]  

(14.41)

For the case of a local central potential, the Born approximation reduces to

\[ \langle \vec{k}_f | T_B | \vec{k}_i \rangle = \langle \phi_{\vec{k}_f} | T_B | \phi_{\vec{k}_i} \rangle = \int d^3r \, \phi^*_{\vec{k}_f} V(r) \phi_{\vec{k}_i} = \frac{1}{(2\pi)^3} \int d^3r \, e^{i\vec{q} \cdot \vec{r}} V(r) \]  

(14.42)

where \( \vec{q} = \vec{k}_i - \vec{k}_f \) is the momentum transfer in the scattering.

As an example of the application of the Born approximation, let us consider the problem of scattering by a Coulomb potential, and in particular the potential due to the nucleus with charge \( Ze \), i.e.

\[ V_C(r) = -\frac{Ze^2}{r} \]  

(14.43)

Here the Born approximation is given by

\[ \langle \phi_{\vec{k}_f} | T_B | \phi_{\vec{k}_i} \rangle = -\frac{Ze^2}{(2\pi)^3} \int d^3r \, e^{i\vec{q} \cdot \vec{r}} \frac{1}{r} \]

\(^7\)By high energy we mean high, relative to the depth of the potential \( V \), or the energy of bound states in the potential \( V \). For example, for electron scattering in atomic physics, high is more than \( 10^3 \, eV \), while for proton scattering in nuclear physics, high is more than \( 10^3 \, MeV \).
This integral, as it stands is not well defined. However, if we replace the Coulomb potential
by the screened Coulomb potential, i.e.,
\[ V(r) = -Ze^2 \frac{e^{-ar}}{r}, \]
the integral becomes well defined, and then we can take the limit as \( a \to 0 \) to get the
Born amplitude for the Coulomb potential, i.e.\(^8\)

\[
\langle \phi_{k_f}|T_B|\phi_{k_i}\rangle = \frac{-Ze^2}{2\pi^2} \lim_{a \to 0} q \int_0^\infty dr \sin qr \\
= \frac{-Ze^2}{2\pi^2} \frac{1}{q^2 + a^2} \\
= \frac{-Ze^2}{2\pi^2} \frac{1}{q^2}, 
\]
where the momentum transfer square is given by
\( q^2 = 2k^2(1 - \cos \theta) \),
and \( \cos \theta = \hat{k}_f \cdot \hat{k}_i \), and \( |\vec{k}_f| = |\vec{k}_i| = k \). The scattering amplitude can now be written as
\[ f(k, \theta) = \frac{Ze^2\mu}{\hbar^2} \frac{1}{k^2(1 - \cos \theta)}. \]

This scattering amplitude is real, and to that extent does not satisfy unitarity. This is
true for the Born approximation for any real potential. With this result, we can calculate
the differential cross section, which is given by
\[
\frac{d\sigma}{d\Omega} = |f(k, \theta)|^2 \\
= \frac{Z^2e^4\mu^2}{\hbar^2} \frac{1}{k^4(1 - \cos \theta)^2} \\
= \frac{Z^2e^4}{16E^2 \sin^2(\theta/2)}. 
\]

\(^8\)We have made use of the integral
\[ \int_0^\infty dx e^{-ax} \sin qx = \frac{q}{a^2 + q^2} \]
to get the Born amplitude for Coulomb scattering.
which is the Rutherford cross section we got in Eq. (10.85) This cross section goes to infinity when \( \cos \theta = 1 \), i.e., for \( \theta = 0 \) and \( \theta = \pi \). In fact, the total cross section is infinity. This is the result of the fact that the Coulomb potential is infinite in range. In nature, we always have screened potentials in the sense that proton-proton scattering is a problem of two charge particles when the two hydrogen atoms overlap. For large distances we have neutral hydrogen atoms. Although the Born amplitude for the Coulomb potential is real and does not satisfy unitarity, the cross section we get is the exact classical Rutherford cross section.

### 14.3 Electron Atom scattering

As a second application of the Born approximation, let us consider electron atom scattering, and in particular, electron scattering from hydrogen. This approximation is very good at high energies where the second term in the Born series, which goes as \( \frac{1}{E} \) becomes negligible. The potential that describes the interaction between the incident electron and the target atom is the sum of two terms. The first term describes the interaction of the electron with the point nucleus, is attractive, and of the form

\[
V_1(r) = -\frac{Z}{r},
\]

where \( Z \) is the charge on the nucleus. Here we are using atomic units. The second term corresponds to the interaction of the incident electron with the charge distribution resulting from the bound electrons in the atom. This is of the form

\[
V_2(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|},
\]

where \( \rho(\mathbf{r}') \) is the charge distribution of the bound electrons and is given by

\[
\rho(\mathbf{r}) = \sum_{n=1}^{Z} |\phi_n(\mathbf{r})|^2.
\]

Here, we note that the sum runs over the \( Z \) electrons each in a state \( \phi_n(\mathbf{r}), n = 1, \ldots, Z \).

For the case of electron hydrogen scattering, \( Z = 1 \) and the bound state electron wave function can be taken as the ground state of the hydrogen atom, i.e., \( \phi_{100}(\mathbf{r}) \). In this case \( V_2 \) is given by

\[
V_2(\mathbf{r}) = \int d^3r' \frac{|\phi_{100}(\mathbf{r}')|^2}{|\mathbf{r} - \mathbf{r}'|},
\]

with

\[
\phi_{100}(\mathbf{r}) = R_{10}(r) Y_{00}(\hat{r}) = \frac{1}{\sqrt{4\pi}} R_{10}(r),
\]