Acknowledgements:

In addition to the classic standard text originally by Herbert Goldstein and now by Goldstein, Poole and Safko (third edition) these notes are derived from my experience listening to, and reading the explanations of others including:

(i) Lecture notes and discussions with Prof. Charles Chew and Prof. Darrell Howe while here at Ohio University.

(ii) A preliminary manuscript for a textbook on a modern approach to Classical Mech. by Jorge José, Northeastern U., Boston MA.


(iv) Modern treatments including "Classical Mech", Metzner & Shepley [Binet's hall (1982)]

Section 1: Foundations of Classical Mechanics

- Brief review of Newtonian Mechanics
- Examples and application including the use of portraits of motion in phase space

Idealization of experimental observations (of non-relativistic systems) leads to the following assumptions of the foundations of Classical Mech:

Foundations of non-relativistic classical mechanics:

I. Kinematics:

Space and time

- Space is three-dimensional and Euclidean (flat)
- Time is one-dim., uniformly and monotonically inc.
Given a reference frame and coordinate system, the position of each point in space can be uniquely represented by three real numbers (coordinates) \((x_1, x_2, x_3)\).

Thus, a position vector can be assigned for any point in space:

\[
\mathbf{r} = x_1 \mathbf{\hat{e}}_1 + x_2 \mathbf{\hat{e}}_2 + x_3 \mathbf{\hat{e}}_3 = \sum_{i=1}^{3} x_i \mathbf{\hat{e}}_i
\]

where \(\mathbf{\hat{e}}_i\) \((i=1, 2, 3)\) are unit vectors forming a basis.

We will use the Einstein summation convention of automatically summing over repeated indices unless otherwise indicated. Thus we can write

\[
\mathbf{r} = x_i \mathbf{\hat{e}}_i \quad \text{(Summation convention)}
\]

Time can be represented by a single real number, \(t\).

Successive positions taken by a point particle as it moves can be parameterized by the time \(t\):

\[
\mathbf{r}(t) = x_i(t) \mathbf{\hat{e}}_i(t)
\]

Note: If the reference frame uses cartesian coord, then the \(\mathbf{\hat{e}}_i(t)\) are constant \(\equiv \mathbf{\hat{e}}_i\).

Note: Time will be quantified more carefully when we discuss dynamics below.
Kinematics of a single particle:

Description of the motion of a single particle:

As time progresses, the position of the particle will trace out a trajectory in space.

Note that the relative shape of the trajectory itself is independent of the particular coordinate system used to describe it. (The origins of the various coordinate systems considered in this discussion are at rest rel to each other.)

The position vector \( \mathbf{r}(t) \) depends on origin of the coordinate system but \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \) are intrinsic to the motion of the particle and independent of the origin or type of coord. system used. (Of course, the components of \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \) in a given coord. system will depend on that system.)

Thus, \( \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{x}'(t) \) and \( \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} \) are properties of the trajectory itself and not those of a particular coord. system.

This geometric property of motion can be emphasized by expressing \( \mathbf{v} \) and \( \mathbf{a} \) in terms of coordinates tied explicitly to the trajectory (called Frenet coordinates).
Digression (Skip on first reading)

Note: we assume the trajectory is smooth (differentiable) both w.r.t space coords. and time.

Note: Really should say that we can always split the time into time intervals where $ds(t) > 0$ (possibly $ds = 0$ only at the end-points of the time intervals)

Then the derivation on pg 14 is rigorous within each time interval.

The main requirement is that $S(t)$ must be an invertable function over the time interval.

Note: A way to define the center of curvature $C$, at time $t$ is to put it in the plane of the three points $\mathbf{r}(t), \mathbf{r}(t+\Delta t)$, and $\mathbf{p} = \mathbf{r}(t+\frac{1}{2}\Delta t)$. Actually $\mathbf{p}$ could be any point on the trajectory between $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$.
Frenét coordinates:

A trajectory of a particle moving from A to B

velocity \( \mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \) \hspace{1cm} (I.1)

Let \( d\mathbf{r} = ds \cdot \hat{t} \) where

\[ ds = |ds| = \text{scalar distance moved in time } dt \]

and \( \hat{t} = \text{unit vector tangent to trajectory (and pointing in the direction of the motion)} \).

Thus \( \{\text{total distance moved from time } t_A \text{ to } t_B\} = \int_{t_A}^{t_B} \frac{ds}{dt} \, dt \) \hspace{1cm} (not net displacement)

Note:
1) Both \( \frac{ds}{dt} \) and \( \hat{t} \) depend on \( s(t) \) and hence on \( t \).
2) \( s(t) \) is monotonic non-decreasing function of \( t \).

Thus, \( \mathbf{r}(t) = \mathbf{r}(s(t)) \)

and \( \frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \cdot \frac{d\mathbf{r}}{ds} \Rightarrow \frac{d\mathbf{r}}{ds} = \hat{t} \)

\[ \therefore \text{velocity } \mathbf{v} = \frac{ds}{dt} \hat{t} = \mathbf{v}(t) \hat{t} \] \hspace{1cm} (I.3)

Also acceleration \( \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}(t) \hat{t} + \mathbf{v} \frac{d\hat{t}}{dt} \) \hspace{1cm} (I.4)

What is \( \frac{d\hat{t}}{dt} \)?

i) \( \frac{d\hat{t}}{dt} \) is \( \perp \) to \( \hat{t} \)

Proof: \( \hat{t} \cdot \hat{t} = 1 \Rightarrow \frac{d(\hat{t} \cdot \hat{t})}{dt} = 0 = 2\hat{t} \cdot \frac{d\hat{t}}{dt} \)

ii) \( \left| \frac{d\hat{t}}{dt} \right| = \mathbf{v} \cdot \frac{1}{\rho} \) where \( \rho \) = radius of curvature

Proof: See diagram at left.

\[ \left| \frac{d\hat{t}}{dt} \right| = \lim_{\Delta t \to 0} \left| \frac{\Delta \hat{t}}{\Delta t} \right| = \lim_{\Delta t \to 0} \left| \frac{\Delta s}{\Delta t} \right| \frac{\Delta \hat{t}}{\Delta t} = \left| \frac{ds}{dt} \right| \frac{d\hat{t}}{ds} \]

From diagram:

\[ \left| \frac{\Delta \hat{t}}{\hat{t}} \right| = \frac{\Delta s}{\rho} \Rightarrow \left| \frac{\Delta \hat{t}}{\hat{t}} \right| = \frac{\Delta s}{\rho} \text{ or } \frac{\Delta \hat{t}}{\Delta s} = \frac{1}{\rho} \]
Thus

\[ \frac{d\hat{\mathbf{E}}}{ds} = \frac{1}{\rho(s)} \hat{\mathbf{N}} \]  

(\text{I.5})

\[ |\frac{d\hat{\mathbf{E}}}{dt}| = \frac{1}{\rho(s)} \frac{d\hat{\mathbf{E}}}{ds} = \nu \cdot \frac{1}{\rho} \]  

(where \( \rho \) is the radius of curvature of the trajectory)

and

\[ \frac{d\hat{\mathbf{E}}}{dt} = \nu \cdot \frac{1}{\rho} \hat{\mathbf{N}} \]  

(\text{I.5a})

Eq. I.5 (or I.5a) essentially defines the unit vector \( \hat{\mathbf{N}} \) as the unit vector \( \perp \hat{\mathbf{E}} \) and directed inward toward the ctr. of curvature. (\( \hat{\mathbf{N}} \) is called the principal normal)

Now in terms of \( \hat{\mathbf{E}} \) and \( \hat{\mathbf{N}} \)

\[ \mathbf{a} = \frac{d\nu}{dt} \hat{\mathbf{E}} + \nu^2 \hat{\mathbf{N}} \]  

accel. tangent centripetal accel. \perp to trajectory  

(\text{I.6})

Also note:

The acceleration vector thus lies in the plane formed by \( \hat{\mathbf{E}} \) and \( \hat{\mathbf{N}} \) called the osculating plane (the instantaneous plane of the trajectory).

The unit vector \( \hat{\mathbf{B}} \) normal to the osculating plane (called the binormal vector) is defined by

\[ \hat{\mathbf{B}} = \hat{\mathbf{E}} \times \hat{\mathbf{N}} \]  

(\text{I.7})

It is left as an exercise to show that \( \frac{d\hat{\mathbf{B}}}{ds} \) (and \( \frac{d\hat{\mathbf{B}}}{dt} \)) is parallel to \( \hat{\mathbf{N}} \).

The "twist" or "torsion" \( \beta(s) \), of the trajectory is defined by

\[ \frac{d\hat{\mathbf{B}}}{ds} = -\beta \hat{\mathbf{N}} \]  

(\text{I.8})  

\[ \Rightarrow \frac{d\hat{\mathbf{B}}}{dt} = -\nu \beta \hat{\mathbf{N}} \]  

Note: \( \beta > 0 \) if moving along RH screw threads.

It can also be shown that

\[ \frac{d\hat{\mathbf{N}}}{ds} = -\frac{1}{\rho} \hat{\mathbf{E}} + \hat{\mathbf{B}} \]  

(\text{I.9})

or

\[ \frac{d\hat{\mathbf{N}}}{dt} = -\nu \frac{1}{\rho} \hat{\mathbf{E}} + \nu \beta \hat{\mathbf{B}} \]  

(\text{I.9a})

Note: Eqs. I.5, I.8 and I.9 are the Frenet-Serret formulas in differential geometry.
Summary of Kinematics using Frenet coordinates:

\[ d = \text{distance traveled measured along the path of the trajectory} \]

\[ \text{Speed} = \frac{d}{dt} \]

Frenet coordinates are based on the orthogonal unit-vector triad \((\widehat{\text{T}}(s), \widehat{\text{N}}(s), \widehat{\text{B}}(s))\) that depends on \(s\).

\( \widehat{\text{T}}(s) \) is tangent vector - is tangent to the trajectory at pt \(s\).

\( \widehat{\text{N}}(s) \) is normal vector - \( \widehat{\text{N}}(s) \) is \( \perp \) trajectory, directed toward ctr. of curvature.

\[ \frac{d\widehat{\text{T}}}{ds} = K(s)\widehat{\text{N}}(s) \quad K(s) = \text{curvature of trajectory at point } s \]

\[ K(s) = \kappa(s) \quad \rho(s) = \text{radius of curvature} \]

\( \widehat{\text{B}}(s) \) is binormal vector - \( \perp \) to both \( \widehat{\text{T}}(s) \) \& \( \widehat{\text{N}}(s) \)

\( \widehat{\text{B}} = \widehat{\text{T}} \times \widehat{\text{N}} \)

\[ \frac{d\widehat{\text{B}}}{ds} = -\beta(s)\widehat{\text{N}}(s) \quad \beta(s) \text{ is called "twist" or "torsion"} \]

\[ \mathbf{s} \leftrightarrow t \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \frac{d\mathbf{r}}{ds} = v \frac{d\mathbf{r}}{ds} \]

\[ \mathbf{V} = \frac{d\mathbf{r}}{dt} = v \widehat{\mathbf{T}} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{dv}{dt} \widehat{\mathbf{T}} + \frac{d^2\mathbf{r}}{ds^2} \widehat{\mathbf{B}} \]

Also, \((\widehat{\text{T}}, \widehat{\text{N}}, \widehat{\text{B}})\) satisfy the Frenet-Serret formulas:

\[ \begin{bmatrix} \frac{d\widehat{\text{T}}}{ds} \\ \frac{d\widehat{\text{N}}}{ds} \\ \frac{d\widehat{\text{B}}}{ds} \end{bmatrix} = \begin{bmatrix} K(s) \widehat{\text{N}}(s) \\ -K(s)\widehat{\text{T}}(s) + \beta(s) \widehat{\text{B}}(s) \\ -\beta(s)\widehat{\text{N}}(s) \end{bmatrix} \]

Thus, given well-behaved curvature \(K(s) = \kappa(s)\) and twist \(\beta(s)\) functions, then \((\widehat{\text{T}}, \widehat{\text{N}}, \widehat{\text{B}})\) are determined for all \(s\) by the Frenet-Serret formulas and an initial condition \(\widehat{\text{T}}(s_0) = \widehat{\text{T}}_0, \quad \widehat{\text{N}}(s_0) = \widehat{\text{N}}_0, \quad \widehat{\text{B}}(s_0) = \widehat{\text{B}}_0\).
An important result from the mathematical study of Differential Equations is very useful in Classical Mechanics:

Existence and Uniqueness theorem:
(stated here without proof)

The differential equation \( \frac{dy(t)}{dt} = f(y,t) \)

with initial condition \( y(t_0) = y_0 \)
has exactly one solution on the interval \([t_0, \infty)\)
if \( f \) is continuous in \( t \) and Lipschitz continuous in \( y \) as long as \( y(t) \) stays bounded.

A function \( f(y) \) is Lipschitz continuous within domain \( D \) if there exists a \( K \geq 0 \) such that,
for all \( y_1, y_2 \) in \( D \)
\[
|f(y_1) - f(y_2)| \leq K |y_1 - y_2|.
\]

Note: A differentiable function is Lipschitz continuous iff it has bounded first derivative.

Physicists call such functions "smooth and well behaved".

\[ \frac{dy}{dt} = f(y,t) \text{ w. I.C. } y(t_0) = y_0 \text{ has a unique solution}
\]
if \( f \) is "smooth and well behaved".
II. Dynamics (Fundamentals)

Experimental observations (again of non-relativistic systems) lead to the assumption of Galilean relativity. That is, the Universe appears to behave exactly the same if all of it were to be:

(i) translated by a fixed vector in space \((\vec{r} \rightarrow \vec{r} + \vec{R})\),
(ii) translated by a fixed time \((t \rightarrow t + \Delta t)\),
(iii) rotated by a fixed angle about any axis \((\vec{r} \rightarrow \vec{R}\vec{r}\vec{R}^{-1})\),
(iv) a constant velocity is added to every point \((\vec{v} \rightarrow \vec{v} + \vec{V})\)

These observations lead to the definition of inertial reference frames.

Also, observations suggest that:

The initial state of a mechanical system (the totality of the positions and velocities of all its points at some instant of time) uniquely determines its state for all time. (Newton's principle of determinacy)

Note:

Newton's three laws are consistent with these observations. For example, Newton's second law

\[
\ddot{\vec{r}} = \mathbf{F}(\vec{r}, \dot{\vec{r}}, t)
\]

has a unique solution for given initial conditions assuming \(\mathbf{F}\) is a sufficiently smooth function of \(\vec{r}, \dot{\vec{r}},\) and \(t\).
Two Fundamental Principles of Classical Mech. (equiv. to Newton's Laws)

The laws of mechanics (equivalent to Newton's Three Laws of Motion) can be formulated in two principles. (Following José closely)

Principle 1: There exist frames of reference, called inertial, with the following two properties.

Property A: Every isolated particle moves in a straight line.

Property B: If the motion of time is quantified by defining the unit of time such that one particular isolated particle moves at constant velocity in this frame, then every isolated particle moves at constant velocity in this frame.

Note Principle 1 quantifies the definition of time and is equivalent to the statement of Newton's first law.

Principle 2: Consider two pels 1 and 2 isolated from all other matter. The two pels are observed from an inertial frame and may interact with each other. Let \( v_j(t) \) be the velocity of pel \( j \) at time \( t \).

There exists a constant scalar quantity \( \mu_{12} > 0 \) and a constant vector \( \mathbf{t} \) such that

\[
\mathbf{v}_1(t) + \mu_{12} \mathbf{v}_2(t) = \mathbf{t}
\]

for all \( t \). Furthermore, \( \mu_{12} \) depends only...
on the pcls 1 & 2 and not on the initial conditions or the inertial frame of the motion while the vector \( \mathbf{IK} \) (although constant throughout motion observed in a given frame) does depend on the initial conditions and the particular inertial frame chosen to view the motion.

Similar experiments performed with pcd 2 and a third pcd 3 and with pcd 3 and pcd 1 yield similar results:

\[
\begin{align*}
\mathbf{V}_2(t) + \mu_{23} \mathbf{V}_3(t) &= \mathbf{L} \\
\mathbf{V}_3(t) + \mu_{31} \mathbf{V}_1(t) &= \mathbf{M}
\end{align*}
\]

As before, the constant vectors \( \mathbf{L} \) and \( \mathbf{M} \) depend on the particular experiment and inertial frame but the \( \mu_{ij} > 0 \) do not.

Finally, completing the statement of Principle 2, the \( \mu_{ij} \) are related such that

\[
\mu_{12} \mu_{23} \mu_{31} = 1.
\]

Note: Principle 2 is equivalent to a statement of the existence of inertial mass (related to the \( \mu_{ij} \)) and a statement of conservation of momentum in the form of Newton's Third Law.
Let $\mu_{ij} = \frac{m_j}{m_i}$ where $m_i$ is an intrinsic property of the $i$-th particle (called inertial mass).

Then $M_{12} M_{23} M_{31} = \frac{M_2}{M_1} \frac{M_3}{M_2} \frac{M_1}{M_3} = 1$ satisfying eqn II.3

and equations II.1) and II.2) become

\[\begin{align*}
& \quad m_1 v_1(t) + m_2 v_2(t) = P_{12} \\
& m_2 v_2(t) + m_3 v_3(t) = P_{23} \\
& m_3 v_3(t) + m_1 v(t) = P_{31}
\end{align*}\]

where $P_{ij}$ are called total linear momenta.

Note the masses $m_i$ are not unique. The $\mu_{ij}$'s are determined from experiment, and the $\mu_{ij}$ are the ratios of masses.

In practice a standard mass is chosen and all other masses are measured relative to it.

Taking the time derivative of eqn II.5 gives

\[\begin{align*}
& m_1 a_1(t) + m_2 a_2(t) = 0 \\
& m_2 a_2(t) + m_3 a_3(t) = 0 \\
& m_3 a_3(t) + m_1 a_1(t) = 0
\end{align*}\]

These eqns. are equivalent to eqns II.5 (or II.1-3) and are the familiar Newton's third law if we define force by $F = ma$. 
Thus we now define the force acting on a particle as

\[ F = ma = m \frac{d^2 r}{dt^2} \]  \hspace{1cm} (4.7)

This is Newton's Second Law.

For the two particles (otherwise isolated from the rest of the universe) the change of velocity of pel 1 is due to the force of interaction with pel 2. That is the force on pel 1 due to pel 2 is \( F_{21} = ma_1 \), similarly \( F_{12} = ma_2 \) is the force on pel 2 due to pel 1 and the first of eqn II.6 is

\[ F_{21} + F_{12} = 0 \]

i.e. Newton's Third Law.

Thus the two principles together with the definition of force is equivalent to Newton's three laws but avoids some logical difficulties such as the problem that Newton's first law seems to simply be a special case of the second law.

The two principles given here are an interpretation of Newton's Laws that was originally due to Mach.
III. Brief review of Newtonian Mechanics

Dynamics of a single particle:

→ Always view motion from an inertial reference frame.

Define Linear Momentum of single pt.: \[ \mathbf{p} = m \mathbf{v} \] (inertial mass defined in principle 2.)

Then Newton's First Law follows from principle 1:

If the particle is isolated (i.e., \( \mathbf{F} = 0 \)) then \[ \mathbf{p} = \text{constant}. \]

(Conservation of momentum for an isolated pt.)

And Newton's Second Law follows from principles 1 & 2 and the definition of Force:

If the particle is not isolated, but is acted on by a force, \( \mathbf{F} \), then

\[ \frac{d\mathbf{v}}{dt} = \mathbf{F} \quad \text{or} \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}. \]

(Since we are only considering the non-relativistic case where \( |\mathbf{v}| < c \) = speed of light and \( m \) is constant.) Note that in a system of particles, the mass of a subsystem may change. We consider this later.

\[ \Rightarrow \quad \text{Thus, the entire problem of single particle classical mech.} \]

is to find the solution of III.2
When the force \( \mathbf{F} \) is a given function of \( \mathbf{r} \), \( \mathbf{v} \), and \( t \), the problem of the motion of a single particle is reduced to finding the solution to the ordinary second-order differential equation:

\[
m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)
\]

subject to initial conditions \( \mathbf{r}(t=0) = \mathbf{r}_0 \) and \( \mathbf{v}(t=0) = \mathbf{v}_0 \). Theorems in differential equations assure us that if \( \mathbf{F} \) is well-behaved the solution exists and is unique. Thus, such a system is called deterministic.

Recently, however, it has become increasingly well-known that even such deterministic systems are not predictable for long times. Predictability is a practical matter requiring calculation. In order to make long time predictions (calculations) of the position of a particle, the stability of the solution is important. That is, how do solutions with nearby initial conditions compare? If their difference, \( |S(\mathbf{r}(t))| \) were to grow exponentially with time then the system turns out to be unpredictable in practice. Such solutions are now called chaotic and we will consider this later.
It is useful to consider other dynamical variables, i.e., other functions of \(\mathbf{r}\) and \(\mathbf{v}\) (or \(\mathbf{r}\) and \(\mathbf{p}\)).

**Definition of angular momentum about point 0:**

\[
\mathbf{L}_0 = \mathbf{r} \times \mathbf{p}
\]

Def. torque about point 0:

\[
\mathbf{N}_0 = \mathbf{r} \times \mathbf{F}
\]

Then it follows from Newton's 2nd law that

\[
\frac{d\mathbf{L}}{dt} = \mathbf{N}.
\]

(The subscript indicating the particular pt. 0 has been dropped since the eqn is true for any \(O\) (must use same pt. 0 for both \(\mathbf{L}_0\) and \(\mathbf{N}_0\), of course.)

Proof - trivial - starting with Newton's 2nd law:

\[
\frac{d\mathbf{p}}{dt} = \mathbf{F}
\]

Choose any point 0.

\[
\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F}
\]

\[
\frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} - \mathbf{r} \times \frac{d\mathbf{p}}{dt}
\]

\[
\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0} \quad \text{(even if m is not constant) since p is parallel to v, DONE}
\]

**Conservation Theorem for angular momentum of a particle:**

If \(\mathbf{N} = 0\), then \(\mathbf{L} = \text{constant}\).
Note: The following statements are equivalent:

The force is conservative if

(i) \[ \oint_C \mathbf{F} \cdot d\mathbf{r} \] is independent of the path \( C \)

or

(ii) \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \] for all closed paths.

or

(iii) \( \nabla \times \mathbf{F}(\mathbf{r}) = 0 \) at every point \( \mathbf{r} \)

or

(iv) \( \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \)

Equivalence meaning the truth of any one of the above implies the truth of the others.

Stokes' theorem: \[ \oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \]
Definition of Kinetic Energy of a particle:
First define Work done by a force $\mathbf{F}$ when moving from point $A$ to point $B$ along path $C$:

$$W_{AB} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \hat{t} ds$$

Line integral along curve $C$ from point $A$ to $B$.

Thus the total work done on the particle as it moves along its trajectory from time $t_1$ to time $t_2$:

$$W_{12} = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \hat{t} ds$$

Line integral along trajectory from position at time $t_1$ to position at time $t_2$.

Now use Newton's 2nd Law:

$$W_{12} = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} dt = \int_{t_1}^{t_2} m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt$$

(assuming $m$ is constant)

Get different result if it is not

but $\frac{d\mathbf{v}}{dt} \equiv a = \frac{dv}{dt} \hat{v} + \frac{v^2}{c^2} \hat{n}$

special relativity

for example,

$$\frac{d\mathbf{v}}{dt} \cdot \hat{t} = \frac{dv}{dt}$$

since $\hat{v} \perp \hat{t}$

Thus,

$$W_{12} = \int_{t_1}^{t_2} m \frac{dv}{dt} \cdot \mathbf{v} dt = \int_{v(t_1)}^{v(t_2)} \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2} m (v(t_2)^2 - v(t_1)^2)$$

$m$ constant

or

$$W_{12} = T_2 - T_1$$

where $T \equiv$ kinetic energy $= \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}$

This is taken as the definition of kinetic energy.

Work - Energy theorem

Statement of Work - Energy Theorem.

If $F \cdot ds = dT$ is Differential.
If all forces acting on the particle are conservative, i.e., if \( \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} \) is independent of the path \( \gamma \) from \( A \) to \( B \) (depends only on the endpoints), then there exists a scalar function of position \( V(r) \) (potential energy) such that

\[
V(r_B) - V(r_A) = -\int_{r_A}^{r_B} \mathbf{F} \cdot d\mathbf{r} \tag{note minus sign}
\]

How is \( V(r) \) related to \( F(r) \)?

Note: From def. of gradient: \( \nabla V \cdot dr = dV \)

Thus,

\[
\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_{r_A}^{r_B} dV = V_B - V_A = -\int_{r_A}^{r_B} \mathbf{F} \cdot d\mathbf{r}
\]

and hence the force will be conservative if \( \exists V(r) \) s.t., \( F = -\nabla V \). \( V \) = potential energy and vice-versa.

Thus, if all forces are conservative,

\[
W_{12} = \oint_{t_1}^{t_2} \mathbf{F} \cdot d\mathbf{r} = -(V_2 - V_1) = V_1 - V_2
\]

and Work-Energy theorem becomes

\[
V_1 - V_2 = T_2 - T_1
\]
or

\[
T_1 + V_1 = T_2 + V_2
\]

But since \( t_1 \) and \( t_2 \) are arbitrary, we have

Total Energy \( = T + V \) = const. if conservative system

Energy conservation theorem for a particle.
Note: Could have $V(r, t)$ such that

\[ F(r, t) = -\nabla V(r, t) \]

but then

\[ \nabla V \cdot dr = dV \]

Since

\[ \nabla V \cdot dr = \left( \frac{\partial V}{\partial x} \frac{dx}{dr} + \frac{\partial V}{\partial y} \frac{dy}{dr} + \frac{\partial V}{\partial z} \frac{dz}{dr} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k}) \]

\[ = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \]

\[ = dV \]

Thus, if $V(r, t)$

then \[ \mathbf{F} \cdot dr = -dV \]

Actually

\[ \oint_C \mathbf{F} \cdot dr = -\oint_C \nabla V \cdot dr = -\int_{T_1}^{T_2} dV + \int_{t_1}^{t_2} \frac{\partial V}{\partial t} dt \]

or

\[ T_2 - T_1 = V_1 - V_2 + \int_{t_1}^{t_2} \frac{\partial V}{\partial t} dt \]

or

\[ E_2 - E_1 = \int_{t_1}^{t_2} \frac{\partial V}{\partial t} dt \]

and Energy is not conserved. Note: Need to know $F(t)$ before we can do this integral.

Note also that \[ \frac{dE}{dt} = \frac{\partial V}{\partial t} \].
System of Particles - Dynamics

System of particles:

$\mathbf{F}_i = m_i \mathbf{a}_i$

Split forces on each particle:

$\mathbf{F}_i = \mathbf{F}_i^{(e)} + \sum_{j \neq i} \mathbf{F}_{ij}$

Note $\mathbf{F}_{ii} = 0$

$F_{ij}$: force between $i$-th and $j$-th particle acts on $i$. ($F_{ij}$ act on $j$)

Newton's third law (weak form): $F_{ij} = -\mathbf{F}_{ji}$

Define: Total linear momentum:

$\mathbf{P} = \sum_i m_i \mathbf{v}_i$

$\frac{d\mathbf{P}}{dt} = \sum_i \frac{d\mathbf{P}_i}{dt} = \sum_i m_i \mathbf{a}_i$

Using Newton's 2nd law for each particle:

$\sum_i \frac{d\mathbf{P}_i}{dt} = \sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{F}_{ij}$

Thus, we have:

$\frac{d\mathbf{P}}{dt} = \sum_i \frac{d^2 m_i \mathbf{v}_i}{dt^2} = \mathbf{F}^{(e)}$

Thus, have total linear momentum conservation theorem:

If $\mathbf{F}^{(e)} = 0$ (isolated system) then $\mathbf{P}_{\text{total}} = \text{constant}$.
Define center of mass position $\mathbf{R}_{cm}$:

$$M\mathbf{R}_{cm} = \sum_i m_i \mathbf{r}_i$$

where $M = \text{total mass} = \sum m_i$.

Thus we have

$$\mathbf{P} = M \frac{d\mathbf{R}_{cm}}{dt} = M \mathbf{V}_{cm}$$

and

$$\frac{d\mathbf{P}}{dt} = M \frac{d^2\mathbf{R}_{cm}}{dt^2} = M \frac{d\mathbf{V}_{cm}}{dt} = M \mathbf{a}_{cm} = \mathbf{F}^{(e)}$$

This is particularly useful in describing the motion of rigid bodies.

Define total angular momentum of a system of particles:

$$\mathbf{L} \text{ total} = \sum_i \mathbf{L}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

$$\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i) = \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} + \sum \mathbf{r}_i \times \frac{d\mathbf{P}}{dt}$$

$$= \sum_i \mathbf{r}_i \times \left( \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ij} \right) = \mathbf{N}^{(e)} + \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij}$$

Proof: $\sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij} = \frac{1}{2} \sum_{i,j} (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ij})$

$$= \frac{1}{2} \sum_{i,j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0$$

(by Strong form of Newton's 3rd law)

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \text{ and act along line joining the points } i, j$$

both forces have same moment arm
Thus we have
\[ \frac{dL}{dt} = N^{(e)} \quad (\text{applied external torque}) \]

**Conservation of total angular momentum:**

If \( N^{(e)} = 0 \) then \( L_{\text{total}} = \text{constant} \).

Now bring in the center of mass: \( M R_{\text{cm}} = \sum m_i \mathbf{r}_i \)

\[ \mathbf{r}_c = R_{\text{cm}} + \mathbf{r}_i \]

\[ \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{dR_{\text{cm}}}{dt} + \frac{d\mathbf{r}_i}{dt} = \mathbf{v}_{\text{cm}} + \mathbf{v}_i \]

\[ L = \sum_i L_i = \sum_i m_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i m_i \mathbf{r}_c \times \mathbf{v}_{\text{cm}} + \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i \]

\[ = M R_{\text{cm}} \times \mathbf{v}_{\text{cm}} + \sum_i (\mathbf{r}_i \times m_i \mathbf{v}_i) + \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i \]

\[ \sum_i m_i \mathbf{v}_i = \frac{d}{dt} \sum_i m_i \mathbf{v}_i = 0 \]

\[ R_{\text{cm}} \times \frac{d}{dt} \sum_i m_i \mathbf{r}_i = 0 \]

\[ L_{\text{total}} = R_{\text{cm}} \times MV_{\text{cm}} + \sum_i R_i \times \mathbf{p}_i \]

**Note:** \( L_{\text{total}} \) depends on \( O \) only through the position of the center of mass with respect to \( O \).
For system of particles:

\[ L_{\text{total}} = L_{\text{CM}} + L_{\text{CM}} \]

where \( L_{\text{CM}} \equiv R_{\text{CM}} \times MV_{\text{CM}} = R_{\text{CM}} \times \mathbf{F}_{\text{total}} \) is angular momentum of CM about \( O \) of particles moving relative to CM taken about CM.

We already have shown that (Assuming strong form of Newton's 3rd)

\[ \frac{dL_{\text{total}}}{dt} = N_{\text{CM}}^{(e)} = \text{total applied (external) torque about } O. \]

Now we show further that

1. \[ \frac{dL_{\text{CM}}}{dt} = R_{\text{CM}} \times \mathbf{F}_{\text{total}}^{(e)} \]

   \[ \text{torque about } O \text{ due to total external force placed at CM.} \]

   and

2. \[ \frac{dL_{\text{CM}}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = N_{\text{CM}}^{(e)} = \text{total torque due to external forces taken about CM.} \]

These are very useful results!!

Proof of 1 is trivial:

\[ \frac{dL_{\text{CM}}}{dt} = R_{\text{CM}} \times \frac{d}{dt} MV_{\text{CM}} + R_{\text{CM}} \times \frac{d}{dt} \mathbf{F}_{\text{total}} = R_{\text{CM}} \times \mathbf{F}_{\text{total}}^{(e)} \]

Proof of 2 is only slightly more work:

\[ \frac{dL_{\text{CM}}}{dt} = \sum_i \frac{d}{dt} m_i \mathbf{r}_i \times m_i \mathbf{v}_i^{(e)} + \sum_i \mathbf{r}_i \times m_i \frac{d\mathbf{v}_i}{dt} \]

\[ \sum_i \frac{d}{dt} m_i \mathbf{v}_i^{(e)} = \frac{dMV_{\text{CM}}}{dt} \]

\[ \frac{d}{dt} \mathbf{F}_{\text{total}} = \frac{d}{dt} \mathbf{F}_{\text{CM}} \]
Thus

\[
\begin{align*}
\frac{d\mathbf{L}_{\text{cm}}}{dt} &= \sum_i \mathbf{r}_i' \times m_i \frac{d\mathbf{v}_i'}{dt} = \sum_i \mathbf{r}_i' \times m_i \frac{d\mathbf{v}_i'}{dt} - \left( \sum_i m_i \mathbf{v}_i \right) \times \mathbf{F}_{\text{cm}}^\text{acc.} \\
&= \sum_i \mathbf{r}_i' \times \frac{d\mathbf{P}_i'}{dt} = \sum_i \mathbf{r}_i' \times \mathbf{F}_i = \sum_i \mathbf{v}_i' \times \mathbf{F}_i^{(e)} + \sum_i \mathbf{v}_i' \times \mathbf{F}_{\text{cm}}^{(e)} \\
&= 0
\end{align*}
\]

by summing over all internal forces.

Thus, change in total angular momentum about the CM is equal to total external torque about CM. (Even if CM is accelerating, i.e., even if CM frame is a non-inertial frame.)
Work and Kinetic Energy for a System of Particles:

**Define:** Configuration space: The set of coordinates which uniquely determines the position of every particle. (If there are N particles, then we need 3N variables. In this case we have a 3N-dimensional configuration space.)

The system is represented by a point in configuration space. A configuration is given by \[ \Pi_1, \Pi_2, \ldots, \Pi_N \equiv \Xi \Pi_c \]

**Define:** Total work done on system in going from configuration 1 to configuration 2:

\[ W_{12} = \sum_i \int_C \mathbf{F}_i \cdot d\mathbf{r}_i \quad \text{where} \quad \mathbf{F}_i = \mathbf{F}_{ei} + \sum_j \mathbf{F}_{ij} \]

where \( C \) is the trajectory followed by the system in the 3N-dimensional configuration space.

**Work-Energy Theorem for system of particles:**

\[ W_{12} = T_2 - T_1 \quad \text{where} \quad T = \sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \]

\( T = \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \) is defined as the Total Kinetic Energy.

**Proof:** Left as exercise for student - follows directly from Newton's 2nd law applied to each particle.
Work-Energy theorem goes over to system of particles with $T$ replaced with total kinetic energy of system:

$$W_{12} = T_2 - T_1$$

where $W_{12}$ = total work done by all forces and

$$T = \text{total kinetic energy} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i v_i \cdot v_i$$

Note: If system of particles forms a rigid body — then internal forces do no work. Then $W_{12} =$ work done by external forces.

**Total kinetic energy in terms of center of mass coordinates:**

$$T = \frac{1}{2} \sum m_i v_i \cdot v_i = \frac{1}{2} \sum m_i \left( V_{cm} + V_i \right) \cdot \left( V_{cm} + V_i \right)$$

or

$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \sum m_i v_i \cdot v_i \left( \frac{(\sum m_i v_i)^2}{\sum m_i v_i} \right)$$

**Useful Result:**

KE of CM

$$V_{cm}$$

KE of system relative to CM.
Before we go on to consider general constraints, let's review simple applications to rigid bodies.

A) Rigid body
   1) internal forces do no work
   2) \( W_{12} = W_{21} = \Delta T = T_2 - T_1 \)
   3) split motion into motion of CM and rotation about CM.

Use center of mass coord. system results.

If have the very simple situation where axes of rotation always points in same direction

\[
T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \omega^2
\]

- KE of CCM
- KE due to motion of system relative to center of mass.

Example:

Find KE of sphere rotating about fixed axis which touches sphere on its surface with angular velocity \( \omega \).

Method 1: \( T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 \)

- \( V_{cm} = R \omega \)
- \( I_{cm} = \frac{2}{5} MR^2 \)

\[
T = \frac{1}{2} MR^2 \omega^2 + \frac{1}{2} \cdot \frac{2}{5} MR^2 \omega^2 = \frac{1}{2} \left( \frac{7}{5} MR^2 \right) \omega^2
\]

Method 2: \( T = \frac{1}{2} I \omega^2 \)

Find \( I_{AA'} \) using parallel axis theorem:

\[
I_{AA'} = I_{cm} + h^2 M = \frac{2}{5} MR^2 + R^2 M = \frac{7}{5} MR^2
\]
Rigid bodies (cont.)

3) cont. Rigid body dynamics is simplified by considering CM motion + rotation about CM:

Motion of center of mass \( \frac{d\mathbf{P}}{dt} = \mathbf{F}^{(e)} \) \( \Rightarrow \) \( M \frac{d\mathbf{V}_{CM}}{dt} = \mathbf{F}^{(e)} \) \( \Leftarrow \) eqn. of motion for CM

Rotation about center of mass \( \frac{d\mathbf{L}_{CM}}{dt} = \mathbf{N}^{(e)}_{CM} \) \( \Rightarrow \) \( I \frac{d\omega}{dt} = N^{(e)}_{CM} \) \( \Leftarrow \) eqn. of motion for rotation about CM

where \( I_{CM} \) = moment of inertia about axis through CM and \( \perp \) to \( \mathbf{L}_{CM} \)

\( \omega \) = angular vel. directed in direction of \( \mathbf{L}_{CM} \)

Example: Yo-yo:

String wrapped around center cylinder - string does not slip.

Describe motion of yo-yo after release from rest.

Method 1:

Motion of CM:

\[ T \text{ ext. in string} \]

\[ M_{T} A_{CM} = W - T \]

Rotation about CM:

\[ I \alpha = T R_{1} \]

String does not slip \( \Rightarrow \) \( R_{1} \omega = V_{CM} \) \( \Rightarrow \) \( R_{1} \alpha = A_{CM} \)

Gives 3 eqns for 3 unknowns \( A_{CM}, \alpha, T \).

Method 2: Energy Conservation:

\[ \frac{1}{2} MV_{CM}^{2} + \frac{1}{2} I \omega^{2} - M_{g} y = \text{const.} \]

\[ E = \frac{1}{2} (M_{T} + M_{CM}) V_{CM}^{2} - M_{g} y = \text{const.} \]

\[ \frac{dE}{dt} = \frac{1}{2} M_{eff} 2 \frac{dV_{CM} V_{CM} - M_{g} V_{CM}}{dt} \Rightarrow \]

\[ A_{CM} = \frac{M_{g} V_{CM}}{M_{eff}} = \frac{(M_{1} + 2M_{2}) g}{(M_{1} + 2M_{2} + \frac{1}{2}M_{1} + M_{2} R_{2}^{2} R_{1}^{2})} \]
Conservative forces for systems of particles:

Assume internal interaction forces are:

i) two-body forces \( \mathbf{F}_{ij} \equiv \text{force on particle } j \text{ due to particle } i \)

ii) satisfying Newton's 3rd law - strong form

iii) \( \mathbf{F}_{ij} \) depends only on separation between particle \( i \) and particle \( j \)

Then \( \mathbf{F}_{ij} = \mathbf{F}_{ij}(r_{ij}) = \mathbf{F}_{ij}(r) \) where \( r = \| \mathbf{r}_j - \mathbf{r}_i \| \)

iv) internal forces are conservative

Then force functions \( V_{ij}(r) \) s.t. (pair potential for)

\[
(a) \quad \mathbf{F}_{ij}(r) = -\nabla_{r_j} V_{ij}(r) \quad \text{where} \quad r = \| \mathbf{r}_j - \mathbf{r}_i \| = \| \mathbf{r}_j - \mathbf{r}_i \| \\
\text{no summation} \quad r = \mathbf{r}_j - \mathbf{r}_i
\]

Properties of interaction pair potential \( V_{ij} \):

(i) \( V_{ij}(r) = V_{ji}(r) \) \quad \text{Note: } r = \| (\mathbf{r}_j - \mathbf{r}_i) \| \text{ always!} \)

\( \text{i.e. (6) means } V_{ij}(\| \mathbf{r}_j - \mathbf{r}_i \|) = V_{ji}(\| \mathbf{r}_i - \mathbf{r}_j \|). \)

\[\begin{align*}
(b) & \quad \nabla_{\mathbf{r}_j} V_{ij}(r) = \nabla_{\mathbf{r}_j} V_{ji}(r) = -\nabla_{\mathbf{r}_i} V_{ij}(r) = -\nabla_{\mathbf{r}_i} V_{ji}(r) \quad \text{(no sum)}
\end{align*}\]

Property (b) simply states that the interaction potential energy does not change if we just exchange the particles.

Properties (c) can be derived using the following:

\[\nabla_{\mathbf{r}} f(s(\mathbf{r})) = \frac{df}{ds} \nabla_{\mathbf{r}} s(\mathbf{r})\]
and

\[ \nabla_{i\rightarrow j} r = \nabla_{i\rightarrow j} |r_j - r_i| = \nabla_{i\rightarrow j} |r_j - r_i| = \hat{r}_{i\rightarrow j} = \text{unit vector directed from } i \text{ to } j \]

and

\[ \nabla_{j\rightarrow i} r = \nabla_{j\rightarrow i} |r_j - r_i| = \nabla_{j\rightarrow i} |r_j - r_i| = -\hat{r}_{i\rightarrow j} \]

Note that (1), (2), and (3) can easily be derived either from the geometric definition of the gradient or by algebraic means. These derivations are left as student exercises.

Property (c) is verified using (1) to obtain

\[ \nabla_{i\rightarrow j} V_{i\rightarrow j}(r) = \frac{dV_{i\rightarrow j}}{dr} \nabla_{i\rightarrow j} r \]

and then (2) and/or (3).

Now we use properties (a), (b) and (c) to show that the total interaction potential function is given by:

\[ V_{\text{int}}(r_1, r_2, \ldots, r_n) = \frac{1}{Z} \sum_{i<j} V_{i\rightarrow j}(|r_i - r_j|) \]

---

Proof: Must show that \( F_{i\rightarrow k}^{\text{int}} = \nabla_{i\rightarrow k} V_{i\rightarrow k}(\sum r_i^2) \)

\[ \nabla_{i\rightarrow k} V_{i\rightarrow k}(\sum r_i^2) = \frac{1}{Z} \sum_{j \neq k} \nabla_{i\rightarrow k} V_{i\rightarrow j}(r) + \frac{1}{Z} \sum_{i \neq k} \nabla_{i\rightarrow k} V_{i\rightarrow k}(r) \]

\[ = \sum_{j \neq k} \nabla_{i\rightarrow j} V_{i\rightarrow j}(r) = \sum_{j} \frac{F_{i\rightarrow j}}{d_{ij}} \text{ from (c)} \]

Q.E.D.
Thus, if external forces are also conservative then the total potential function is

\[ V(r_1, r_2, \ldots) = \sum_{i} V_{i}^{(e)}(r_i) + \frac{1}{2} \sum_{i,j} V_{ij}(r_i - r_j) \]

Now the total work done if system moves from configuration \( 1 = \{ r_1^{(1)}, r_2^{(1)}, \ldots r_N^{(1)} \} \) to configuration \( 2 = \{ r_1^{(2)}, r_2^{(2)}, \ldots r_N^{(2)} \} \) is given by

\[
W_{12} = -\sum_{i} \int_{1}^{2} \nabla_{r_i} V_{i}^{(e)} \cdot dr_i - \int_{1}^{2} \sum_{i,j} \nabla_{r_i} V_{ij}(r_i - r_j) \cdot dr_i
\]

\[ W_{12} = \left[ W_{12}^{\text{ext}} W_{12}^{\text{int}} \right] \]

Some manipulation gives \( W_{12}^{\text{int}} = -\frac{1}{2} \sum_{i,j} V_{ij}(r_i - r_j) \)

\[
W_{12} = V_1 - V_2 \quad \text{where} \quad V(r) = \sum_{i} V_{i}^{(e)}(r_i) + \frac{1}{2} \sum_{i,j} V_{ij}(r_i - r_j)
\]

The Work-Energy theorem becomes (for conservative system)

\[ T_1 + V_1 = T_2 + V_2 \quad \text{or} \quad T + V = \text{const.} = E \]

E is called the total mechanical energy of system.
Before considering the difficulties in using the Newtonian formulation we consider a few examples and general methods.

One-dimensional motion of single particle.

Assume \( F = F(x) \) conservative \( \Rightarrow V(x) \) s.t. \( F(x) = -\frac{dV}{dx} \)

Total energy is conserved:

\[
T + V = \text{const.} = E
\]

(first integral of the motion)

\[
\frac{1}{2} m v^2 + V(x) = E
\]

\[ v = \pm \sqrt{\frac{2}{m} \sqrt{E - V(x)}} \]

\begin{align*}
\frac{dx}{dt} = \pm \sqrt{\frac{2}{m} \sqrt{E - V(x)}} & \quad \text{first order diff. eqn.} \\
\frac{1}{2} \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - V(x')}} &= \int_{t_0}^{t} dt' = t - t_0 & \quad \text{Problem is reduced to quadrature}
\end{align*}

must separate motion into sections bounded by the turning points where \( \frac{dx}{dt} = 0 \). From one turning point to the next the sign remains unchanged.

For oscillatory motion with turning points \( x_{\text{min}} \) and \( x_{\text{max}} \) we can express the period as

\[
T = 2\sqrt{\frac{x_{\text{max}}}{\frac{2}{m} \sqrt{E - V(x)}}}
\]

Note: must be somewhat careful about the evaluation of the integral since \( E - V(x) = 0 \) at \( x_{\text{min}} \) and \( x_{\text{max}} \).
Thus, the one-dim. conservative force problem can always be "reduced to quadrature" i.e. reduced to a single integral:

\[ t - t_0 = \frac{1}{\sqrt{2m}} \int_{x(t_0)}^{x(t)} \frac{dx'}{\sqrt{E - V(x')}} \]

If we can perform the integration then we obtain the result

\[ t - t_0 = \mathcal{I}(x(t), x(t_0), E) \]

which can be inverted (solved for \( x(t) \)) to get

\[ x = x(t - t_0, x(t_0), E) \]

Even if we cannot carry out the integration, the energy integral is very useful in providing a qualitative description of all possible types of motion for a given system.

For example, from \( \frac{1}{2}mv^2 = E - V(x) \) we can immediately determine ranges of \( x \) that are excluded for classical motion for a given \( E \) since \( E - V(x) = KE \geq 0 \).

A very good general description of motion of a system can be obtained graphically as described below.
Note: Can form alternative statement of the single particle dynamical problem

\[ m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r}, \mathbf{v}) \]

(3 scalar) 2nd order 3x3 eqns

\[
\begin{align*}
\dot{x} &= F_x \\
\dot{y} &= F_y \\
\dot{z} &= F_z
\end{align*}
\]

Can turn this into \([3 \times 2] = 6\) first order 3x3 eqns.

\[
\begin{align*}
\dot{w} &= \mathbf{v} \\
\dot{v} &= \mathbf{F}(\mathbf{r}, \mathbf{v})
\end{align*}
\]

6 first order scalar differential eqns

Solutions are curves in the 6 dimensional vel-phase space
dynamical trajectories

\[
\begin{align*}
x &= x \\
y &= y \\
z &= z \\
v &= v \\
p_x &= p_x \\
p_y &= p_y \\
p_z &= p_z
\end{align*}
\]

vel phase space
phase space

Then have autonomous eqns in this new extended phase space.

If 1-D case

\[
\begin{align*}
\frac{dx}{dt} &= \mathbf{v} \\
\frac{dv}{dt} &= \frac{\mathbf{F}(x, v)}{m}
\end{align*}
\]

Global description of all possible motions is displayed by showing representative trajectories in phase-space called a phase portrait.
Obtaining a global description of the general behavior of a dynamical system: (The velocity phase portrait)

For a conservative system, $V(x)$ exists and $E=U+V=const$. Fixed points and their stability can be obtained from plot of $V(x)$.

Velocity phase space:

Note: for conservative system the orbits in velocity phase space are determined by the total energy $E$ by 

$$\frac{mv^2}{2} + V(x) = E$$

Note the "separatrix" passes thru the hyperbolic (unstable) fixed point and separates the regions of velocity phase space with orbits of a qualitatively different nature.
For conservative system the orbits are given by
\((x, \dot{x})\) that satisfy
\[
\frac{m\dot{x}^2}{2} + V(x) = E \quad \text{eqn. for orbits in vel. phase space.}
\]

\((\text{minima in } V(x)) \Rightarrow \frac{dV}{dx}\big|_{x=x_e} = 0\)

Near the stable fixed point, \(V(x)\) can be expanded
\[
V(x) = V(x_e) + \frac{1}{2} \alpha (x-x_e)^2
\]

where \(\alpha = \frac{d^2V}{dx^2}\big|_{x=x_e} > 0\)

Thus, the vel. phase space orbit eqn becomes
near \(x = x_e\):
\[
\frac{m\dot{x}^2}{2} + \frac{(x-x_e)^2}{2 \left(\frac{1}{\alpha}\right)} = E - V(x_e) \equiv \Delta E > 0 \Rightarrow V(x_e) \text{ is a minimum}
\]

or
\[
\frac{m\dot{x}^2}{2} \Delta E + \frac{(x-x_e)^2}{2 \left(\frac{1}{\alpha}\right) \Delta E} = 1
\]  
\[\text{equation for an ellipse with ctr at } x = x_e \text{ and } \Delta E > 0\]

Similarly, one can show that the orbits in vel. phase space near maxima in \(V(x)\) (i.e. \(x = x_h\)) are hyperbolas.

Thus, \(x = x_e\) fixed point is called elliptic.
and \(x = x_h\) fixed point is called hyperbolic.

---

Note: Eq. for an ellipse, ctr. at \((x_0, y_0)\)
semi-axes at \(a, b\)
\[
\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1
\]
Note that the orbits in velocity phase space are solutions of the differential equations:

\[
\frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = F(x) \cdot \frac{v}{m} = -\frac{1}{m} \frac{dV(x)}{dx}
\]

for a conservative system

Thus, \( \frac{dx}{dv} \mid _{\text{orbit}} = -m \frac{v}{\frac{dV}{dx}} \) if \( \frac{dV}{dx} \neq 0 \).

So \( \frac{dx}{dv} \mid _{\text{orbit}} = 0 \) when \( v = 0 \) (unless \( \frac{dV}{dx} = 0 \)).

Similarly, \( \frac{dv}{dx} = 0 \) when \( \frac{dV(x)}{dx} = 0 \) (unless \( v = 0 \)).

Finally, consider the points where both \( v = 0 \) and \( \frac{dV}{dx} = 0 \) — these are the fixed points mentioned before.

These points are also sometimes called singular points.

We can determine the slope, \( \frac{dv}{dx} \mid _{\text{orbit}} \) on the separatrix at \( x = x_n \): Near \( x = x_n \), \( V(x) = V(x_n) - \frac{1}{2} b (x-x_n)^2 \)

where \( b = -\frac{d^2V}{dx^2} \mid _{x=x_n} > 0 \).

Thus, on the separatrix near \( x = x_n \)

\[
\frac{m}{2} v^2 + V(x_n) - \frac{1}{2} b (x-x_n)^2 = E = V(x_n)
\]

Therefore, \( v = \pm \sqrt{\frac{m}{b}} (x-x_n) \Rightarrow \frac{dv}{dx} \mid _{x=x_n} = \pm \sqrt{\frac{b}{m}} \).
Dissipative systems: (Energy loss)

Now suppose we add a dissipative force to this system.

Suppose there are viscous force \( F_v = -c v \)

Then energy is no longer conserved. Instead the system will tend to move to lower energy as long as \( v \neq 0 \). (Some initial conditions spiral into \( x = x_e \) and others go to \( x \to \infty \).)

The differential equations become:

\[
\frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = -\frac{c}{m} v - \frac{1}{m} \frac{dV}{dx}
\]

\[
\therefore \quad \frac{dv}{dx} \bigg|_{\text{orbit}} = -\frac{1}{m} \left( \frac{c v + \frac{dV}{dx}}{v} \right) \quad \text{if} \quad v \neq 0.
\]

If \( c \) is not too large, the velocity phase portrait will look as follows:

- **fixed point** \( x = x_e \) becomes an attractor (stable spiral or focus)
- shaded region is the "basin of attraction" for fixed point at \( x = x_e \)

Velocity phase space portrait with dissipation.
Look more closely at the phase portrait near the equilibrium (or fixed) points:

First consider the stable equilibrium point \( \text{xe} \).

Linearize the equations of motion about the fixed pt.

\[
V(x) = V(\text{xe}) + \frac{1}{2} a (x - \text{xe})^2 \quad \text{where} \quad a = \frac{d^2 V}{dx^2} \quad \text{effective spring constant}
\]

Let \( x' = x - \text{xe} \)

\[
\frac{dx'}{dt} = \frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = -\frac{c}{m} v - \frac{a}{m} x'
\]

or

\[
\ddot{x} + \frac{c}{m} \dot{x} + \frac{a}{m} x = 0
\]

Has general solution:

\[
x(t) = A e^{-p_1 t} + B e^{-p_2 t}
\]

where

\[
p_1, p_2 = \frac{1}{2} \left[ \frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4 \frac{a}{m}} \right]
\]

Of course,

\[
u(t) = -p_1 A e^{-p_1 t} - p_2 B e^{-p_2 t}
\]

Two cases:

I) \( \frac{c^2}{m^2} < \frac{4a}{m} \) \quad \Rightarrow p_1 \text{ and } p_2 \text{ are imaginary (Re} p_i > 0) \quad \text{underdamped}

Phase portrait \( \rightarrow \) stable spiral (or focus)

II) \( \frac{c^2}{m^2} > \frac{4a}{m} \) \quad p_{1,2} \text{ are real and } 0 \leq p_1 \leq p_2 \quad \text{overdamped}

Phase portrait \( \rightarrow \) stable node
For \( \frac{c^2}{m^2} > \frac{q}{m} \) where \( a = \frac{d^2 \psi}{dx^2} \bigg|_{x = x_e} \):

If \( A = 0, \ B = x_0 \) \( 0 < p < p_2 \)

\[
\begin{align*}
x(t) &= x_0 e^{-p_2 t} \\
v(t) &= -p_2 x_0 e^{-p_2 t} = -p_2 x'
\end{align*}
\]

\[p_2 = \frac{1}{m} \left[ \frac{c^2}{m} - \sqrt{c^2 - q^2} \right] \]

\[p_1 = \frac{1}{m} \left[ \frac{c^2}{m} + \sqrt{c^2 - q^2} \right] \]

\[x \text{ near } t = 0 = x' \]

\[\text{slope} = -p_2 \text{ fast} \]

Similarly if \( A = x_0, \ B = 0 \)

\[
\begin{align*}
x(t) &= x_0 e^{-p_1 t} \\
v(t) &= -p_1 x_0 e^{-p_1 t} = -p_1 x'
\end{align*}
\]

\[p_1 = \frac{1}{m} \left[ \frac{c^2}{m} - \sqrt{c^2 - q^2} \right] \]

\[0 < p_1 < p_2 \]

\[\text{slopes} = -p_1 \text{ slow} \]

For arbitrary initial cond. near \( x = x_e \) \( v = 0 \)

\[\frac{dx}{dv} = 0 \text{ if } x = x_e \]

\[v = 0 \]
Near $x_h$ - linearize eqns:

$$x' = x - x_h$$

$$V = V(x) = \frac{1}{2} b x'^2 \quad (x_h \text{ is local max.})$$

$$\frac{d^2 x}{dt^2} + \frac{c}{m} \frac{dx}{dt} - \frac{b}{m} x = 0$$

$$b = -\frac{d^2 V}{dx^2} > 0$$

Has general solution

$$x' = Ae^{-p_1 t} + Be^{-p_2 t}$$

$$p_1 = \frac{1}{2} \left[ \sqrt{\frac{c}{m} + (\frac{c^2}{m^2} + \frac{4b}{m})^{1/2}} \right]$$

Note that $p_1 \neq p_2$ are always real

and $p_1 < 0$ while $p_2 > 0$ (if $b > 0$)

:. $p_1$ is unstable while $p_2$ is stable.

$A = 0 \Rightarrow B = x_0$

$V = -p_2 x'$; $p_2 > 0$

$A = x_0'$; $B = 0$

$\gamma\rightarrow \gamma = -p_1 x'; p_1 < 0$

Unstable manifold (outset)

$A = x_0'$; $B = 0$

Slope $= -p_1 < 0$

$\gamma\rightarrow \gamma = -p_2 < 0$

Stable manifold (inset)

For arbitrary initial $\gamma$, near $x = x_h$; $V = 0$