Difficulties which arise if simply try to turn the mathematical crank on

\[ \text{N-body problem: } \quad M_i \ddot{r}_i = F_i(r_1, r_2, \ldots) \quad \text{for } i = 1, 2, \ldots, N. \]

1) **Three (or more) body problem**—if have more than two bodies then in general, no analytic solution exists!

2) **Non-linear effects**—again, generally, no analytic solutions! Nearly all "real" problems are non-linear.

Formal treatment of both these problems is difficult and we will delay (until next quarter) their discussion.

3) **Constraints**—constraints place restrictions on the allowed configurations of the system. This causes two difficulties:

**Difficulty (i)** → (i) The coordinates describing the system are no longer independent—hence the equations of motion are not independent.

**Difficulty (ii)** → (ii) The constraints will exert forces on the particles which make up the system. These forces of constraint cannot be specified directly, but become known only after the motion is determined.
Some examples of constraints:

1. Particle constrained to move (slide) on a smooth curved surface, while under influence of gravity.

Note: the force on body consists of two terms

   1. force of gravity — known initially, i.e., specified directly

   2. force of constraint: force surface exerts on the body.
      not known initially, i.e., must know motion of body before can determine.

2. Two rods connected at one end by a hinge, hung from ceiling by a hinge and constrained to move in the plane. (double pendulum)
   - Force of gravity again known
   - but forces at hinges are not known, until motion is found
   - also forces which hold rod together as a rigid rod are not known until motion is known.

Note: Generally, the forces of constraint depend not only on the configuration (position) of the system but also in the velocities of the various parts of the system. These velocities are not known until have solved the problem!
Examples Continued:

3. A bead constrained to move on a circular wire hoop which is rotating about a vertical axis along its diameter.

\[ (r, \theta, \phi) \] coordinates

(moving constraint)

Constraint equations:

- \( r = a = \text{constant} \)
- \( \phi = \omega t \) — explicit time dependence

4. Ball bouncing on a horizontal floor.

Constraint relations: \( x > 0 \) and if \( x = 0 \) then \( x \) is set as \(-x\).

Classification of constraints:

(a) Fixed — holonomic
   or moving — rheonomic — equation of constraint contains time explicitly.

(b) Expressible as \( m \) equations — holonomic

of the form

\[ f_k(\Gamma_1, \Gamma_2, \ldots, \Gamma_n, t) = C_k \quad k = 1, \ldots, m \]

\( C_k \) constant

(of course \( g_k(\Delta \Gamma, t) = 0 \) where \( g_k = f_k - C_k \))

or non-holonomic (inequalities or non-integrable differential constraint)

Example 1

\( r^2 \geq \alpha^2 \) (particle sliding down a sphere from a point near the top under the influence of gravity will eventually fall off the sphere)

Example 2

Gas molecules in box.

Example 3 — Example 4 above (ball bouncing on a horizontal floor)
Example of a general differential equation of constraint:

Suppose coordinates are \((x_1, x_2, x_3)\) with equation of constraint

\[
\sum_{\alpha=1}^{3} A_{\alpha}(x^\alpha, t) \frac{dx^\alpha}{dt} + B(x^3, t) = 0 \quad (\alpha = 1, 2, 3)
\]

or \[
\sum_{\alpha} A_{\alpha} dx^\alpha + B dt = 0.
\]

In general, this equation will not be integrable \(\Rightarrow\) non-holonomic constraint.

But if it is integrable, then we can find a function \(F(x^3, t)\) such that

\[
F(x^3, t) = \text{constant}.
\]

i.e., \[
\frac{dF}{dt} = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \frac{\partial F}{\partial x_3} dx_3 + \frac{\partial F}{\partial t} dt = 0
\]

Thus, the equation will be integrable if

\[
A_{\alpha}(x^3, t) = \frac{\partial F}{\partial x_\alpha} \quad \text{and} \quad B(x^3, t) = \frac{\partial F}{\partial t}.
\]

Example of an integrable differential constraint is a cylinder rolling without slipping on a table.

\[
\begin{align*}
\frac{dx}{dt} &= R \frac{d\theta}{dt} \quad \Rightarrow \quad dx - R d\theta = 0 \\
\int dx &= R \int d\theta \\
x + \text{const} &= R \theta + \text{const} \Rightarrow x - R \theta = \text{const.} \\
(f(x, \theta) &= \text{const.})
\end{align*}
\]
Definitions: The generalized coordinates are a minimal set of quantities necessary to uniquely specify the configuration (i.e., the position of all particles) consistent with the constraints. Note that the set of gen. coord. is not unique, but the number of quantities in the set is unique for a given system.

Velocity

Definition: The number of degrees of freedom is the number of quantities which must be specified to determine all the velocities for any motion which does not violate the constraints. (Roughly stated, it is the number of independent ways the system can be moved consistent with the constraints).

Note: If the constraints are holonomic, then the number of generalized coordinates is equal to the number of degrees of freedom. If non-holonomic constraints, then the number of generalized coord. may be greater than the number of velocity degrees of freedom.
We showed earlier (pg 2-4) that a cylinder rolling without slipping in a straight line is an example of a differential equation of constraint which is integrable and hence holonomic.

However, in general rolling without slipping is not a holonomic constraint, i.e. gives differential e.q.s. of constraint that are not integrable.

The standard example: Consider a thin cylinder, of radius a, rolling without slipping on a horizontal plane such that the plane of the disk is always vertical.

Generalized Coordinates:

\[ x, y, \theta, \phi \]

Constraint equations: rolls without slipping on plane

\[ v = a \frac{d\phi}{dt} = a \dot{\phi} \]

and

\[ x = v \sin \theta \quad \Rightarrow \quad \dot{x} - a \sin \theta \dot{\phi} = 0 \]

\[ y = -v \cos \theta \quad \Rightarrow \quad \dot{y} + a \cos \theta \dot{\phi} = 0 \]

We show below that these two e.q.s. of constraint are not integrable.

Note: In this example we need 4 generalized coord. to specify the configuration. But only need 3 quantities to specify velocities, i.e. 3 degrees of freedom. (For example: \( v_x, v_y \), and \( \frac{d\theta}{dt} \) since we can get \( \dot{\phi} \) from \( a \frac{d\phi}{dt} = \sqrt{v_x^2 + v_y^2} \).) Also note that these three degrees of freedom are not all independent of the gen. coord. For example \( \theta = \arctan \left( \frac{v_x}{v_y} \right) \). Thus, dynamical state space is \( 4 + 2 = 6 \) dim. here.
Classic example of non-integrable constraint continued:

Proof that the differential equations of constraints

\[ dx - a \sin \Theta \, d\phi = 0 \]
\[ dy + a \cos \Theta \, d\phi = 0 \]

are not integrable.

Two approaches:  
1. Mathematical approach -- see Exercise 1-7 Goldstein 2002
2. Physical approach - given below:

Clearly both equations cannot be integrable because if they were we would have two relationships

\[ f_1 (x, y, \Theta, \phi, t) = 0 \]
\[ f_2 (x, y, \Theta, \phi, t) = 0 \]

These two relations could be used to eliminate two of the variables - say \( \Theta \) and \( \phi \) and hence the complete configuration would be uniquely determined by \((x, y)\) alone. But this is clearly not the case since \( \Theta \) can be anything for a given \( x \) and \( y \) consistent with the constraints. Thus, \( \Theta \) must also be given.

Even one of the equations cannot be integrable, because then we could eliminate one variable - say \( \phi \). But can get any \( \phi \) for some \((x, y, \Theta)\) by rolling cylinder around circles of different diameters.
Note:
Suppose you are given a differential constraint eqn.

\[
\sum_{i=1}^{N} \frac{\partial}{\partial x_i} g_i(x_1, t) \, dx_i + \frac{\partial}{\partial t} g_0(x_1, t) \, dt = 0.
\]

How can you quickly determine whether this (as it stands, without any modification) is an exact differential?

Ans. \( \exists \) If it is then

\[
\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}
\]

Proof:
If it is an exact differential then \( \exists \) function \( F(x_i, t) = \text{const.} \)

on
\[
dF = \sum_{i=1}^{N} \frac{\partial F}{\partial x_i} \, dx_i + \frac{\partial F}{\partial t} \, dt = 0
\]

\[
\frac{\partial F}{\partial x_i} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \frac{\partial g_j}{\partial x_i}
\]

\[
= \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{N} \frac{\partial g_j}{\partial x_j} \right) = \frac{\partial}{\partial x_i} (0) = 0
\]

QED
The mathematical approach to showing whether or not a differential constraint is holonomic is based on finding a function (called the integration factor) which will turn the differential constraint equation into an exact differential.

\[ \sum_{i} q_i (x^3, t) \, dx_i + q_t (x^3, t) \, dt = 0 \]

will be holonomic if there exists a function \( f(x^3, t) \) (called the integration factor) which will turn \( Q \) into an exact differential. That is,

\[ f(x^3, t) \left[ \sum_{i} q_i (x^3, t) \, dx_i + q_t (x^3, t) \, dt \right] = 0 \]

can be written as

\[ df(x^3, t) = 0 \]

where

\[ \frac{\partial f}{\partial x_i} = f(x^3, t) \frac{\partial q_i}{\partial x_i} \quad \text{for all } i \]

and

\[ \frac{\partial f}{\partial t} = f(x^3, t) \frac{\partial q_t}{\partial t} . \]

If you can show that it is impossible to find an integration factor function for a particular differential constraint eqn., then it is non-holonomic.

Exercise 1-4 in Goldstein 3rd ed. is to use this method to show the equations of constraint for rolling w/o slipping on a plane (pg. 2-6) are non-holonomic.
Now we discuss techniques which we can use to remove the difficulties which arise because of constraints. (ii) and (ii) (pg. 2-1)

Method I (Generalized Coordinates):

1st general technique — applicable if we have holonomic constraints —

Use the equations of constraints to reduce the no. of coordinates necessary to determine the configuration of the system.

N. pels ⇒ 3N coordinates
m. holonomic eqns. of constraints relating the coordinates

\[ 3N - m \quad \text{independent coordinates} \]

° Introduce 3N - m new, independent coord. \[ \{q\}_3 \]

Obtain 3N - m independent differential equations for motion of the \[ \{q\}_3 \] generalized coord.,

Solve these equations.

Method II (Lagrange multipliers):

2nd general technique —

Keep the original 3N coordinates and add the constraint equations (in differential form) to the system of differential equations; but treat the 3N coordinates as independent by introducing a new unknown multiplier for each differential equation of constraint.

— Method of Lagrange multipliers —
Elimination of Forces of Constraint from problem:

Start the discussion of these two techniques
- generalized coordinates

or

- Lagrange multipliers

by considering the principle of Virtual Work.

Consider a system in static equilibrium.

Then \( F_i = 0 \) for every particle (i.e., for all \( i \)).

Now consider a virtual displacement of the system

where each \( F_i \rightarrow F_i + \delta F_i \) while time is held constant.

The \( \delta F_i \) are taken to be consistent with all constraints.

The virtual work \( SW = \sum_{i=1}^{N} F_i \cdot \delta F_i \).

Clearly, if the system is in equilibrium then \( SW = 0 \),

\[ \sum_{i=1}^{N} F_i \cdot \delta F_i = 0 \quad \text{(equilibrium)} \]

We may split the total force \( F_i = F_i^{(c)} + F_i^{(a)} \) where \( F_i^{(c)} \) are the forces of constraint and \( F_i^{(a)} \) are all other forces.

\( F_i^{(a)} \) is sometimes referred to as the "applied" force on \( i \).

Now, we limit consideration to constraints for which the forces of constraint do no net work during a virtual displacement (note this excludes sliding friction! But rolling without slipping is still included).
Thus, if the forces of constraint do no net work, then
\[
\sum_{i=1}^{N} f_i^{(a)} \cdot s_{i} = 0 \quad \text{(static equilibrium)}
\]

Principle of virtual work.

Note: The forces of constraint have been eliminated from the problem of finding static equilibrium configuration.

In cartesian coordinates: (dropping the superscript \(a\))

\[
F_x = F_{x_i} \hat{x} + f_{y_i} \hat{y} + f_{z_i} \hat{Z} \\
\text{and} \quad S_{ix} = S_{ix_i} \hat{x} + S_{iy} \hat{y} + S_{iz} \hat{Z}
\]

The principle of virtual work becomes

\[
SW = \sum_{i=1}^{N} (F_{x_i} S_{ix_i} + f_{y_i} S_{iy} + f_{z_i} S_{iz}) = 0 \quad \text{(equilibrium)}
\]

This can be written more compactly:

\[
SW = \sum_{j=1}^{3N} F_j S_{x_j} = 0 \quad \text{(equilibrium)}
\]

where the index \(j\) runs over all particles and over directions \(\hat{x}, \hat{y}\) and \(\hat{z}\) for each particle.
Now we consider the two methods mentioned earlier applied to the problem of finding the equilibrium configuration using the principle of virtual work.

**Method I**: (Generalized Coordinates)

If the constraints are holonomic, then we can use a set of $3N - m$ independent generalized coordinates

$$q^i, \quad i = 1, \ldots, 3N - m.$$  

The position vectors $\mathbf{r}_i = \mathbf{r}_i(q^1, \ldots, q^{3N-m}, t)$.

Thus, for any virtual displacement

$$\delta \mathbf{r}_i = \sum \frac{\partial \mathbf{r}_i}{\partial q^j} \delta q^j$$

or (using notation of pg 31)

$$\delta x_j = \sum \frac{\partial x_j}{\partial q^k} \delta q^k.$$  

Hence,

$$\delta W = \sum \delta \mathbf{r}_i \cdot F_i = \sum \delta \mathbf{x}_j \cdot F_j = \sum \frac{\partial \delta \mathbf{x}_j}{\partial q^k} F_j = \sum \frac{\partial \delta x_j}{\partial q^k} F_j = \sum \frac{\partial \delta x_j}{\partial q^k} \frac{\partial x_j}{\partial q^k} F_j = \sum Q_k \delta q_k,$$

where

$$Q_k = \sum F_j \frac{\partial x_j}{\partial q_k} = Q_k(q^1, \ldots, q^{3N-m}, t) = \text{Generalized force associated with } q^k.$$

Note: Not considering velocity dependent forces applied here.

Now, the principle of virtual work applied to static equilibrium gives

$$\sum Q_k \delta q_k = 0 \Rightarrow Q_k = 0 \text{ for all } k \text{ (static equilibrium)}$$

since the $q_k$ and $\delta q_k$ are independent.

This gives the same no. of equations as we have generalized coordinates. Hence solve the $3N - m$ equations for the $3N - m$ unknowns $q_1, \ldots, q_{3N-m}$ which is the static equilibrium configuration.
Method II: (Lagrange multipliers)

System configuration is determined by \(3N\) coordinates \(\{x_j\}\), where \(j = 1, \ldots, 3N\). But all these coordinates are not independent. Assume \(m\) eqns. of constraint:

\[ \Phi_k(\{x_j\}) = 0 \quad \text{where } k = 1, \ldots, m \]

or

\[ \sum_j \frac{\partial \Phi_k}{\partial x_j} s_{x_j} = 0 \quad , \quad k = 1, 2, \ldots, m \]

Now the principle of virtual work gives

\[ \sum_j s_{x_j} = 0 \quad \text{where } s_{x_j} \Phi_k(\{x_j\}) \]

is the work done by "applied" forces (i.e., all forces except forces of constraints) when coordinate \(x_j\) is changed by an infinitesimal \(s_{x_j}\), consistent with the constraints. The \(s_{x_j}\)'s are not independent because of the constraint equations. Thus, cannot set each

\[ s_{x_j} = 0 \]

Method of Lagrange Multipliers:

Introduce \(m\) new parameters which are unknown, \(\lambda_k\).

Then adjust these so that the \(3N\) coordinates can be treated as independent:

\[ \sum_{j=1}^{3N} F_j s_{x_j} + \sum_{k=1}^{m} \lambda_k \sum_{j=1}^{3N} \frac{\partial \Phi_k}{\partial x_j} s_{x_j} = 0 \]

or

\[ \sum_{j=1}^{3N} \left( F_j + \sum_{k=1}^{m} \lambda_k \frac{\partial \Phi_k}{\partial x_j} \right) s_{x_j} = 0 \]
Method II continued:

Now can treat the $Sx_j$ s. as independent:

\[ F_j + \sum_{k=1}^{m} \lambda_k \frac{\partial \phi_k}{\partial x_j} = 0 \quad \text{for } j = 1, \ldots, 3N \ (3N \ \text{equations}). \]

Together with the equations of constraint:

\[ \phi_k(Sx_j) = 0 \quad \text{for } k = 1, 2, \ldots, m \ (m \ \text{eqns}). \]

This gives $3N + m$ equations which can be solved for the $3N + m$ unknowns: $\{Sx_j\}$ and $\{\lambda_k\}$ to find the static equilibrium point.

Note also that the physical significance of the $\lambda_k$ is apparent since at the equilibrium point the applied forces $F_j$ must be balanced by the constraint forces. Thus,

\[ F_j + F_j^{(c)} = 0 \]

Thus, from (7) above,

\[ F^{(c)}_j = \sum_{k=1}^{m} \lambda_k \frac{\partial \phi_k}{\partial x_j} \quad \text{Force of constraint along direction } Sx_j. \]
Example 1

Position of c.m. = x, y

and angle \( \alpha \)

Constraint equations:

\[ y = \frac{l}{2} \sin \alpha + b \sin 30^\circ \text{ where } b = \frac{L}{2} \cos (\alpha + 30^\circ) \]

or

\[ b = \frac{L}{2} (\cos \alpha \cos 30^\circ - \sin \alpha \sin 30^\circ) \]

\[ x = b \cos 30^\circ - \frac{l}{2} \cos \alpha \]

or

\[ b = \frac{l}{2} (\sqrt{3} \cos \alpha - \frac{1}{2}) \]

\[ \text{1. General coords: } \alpha \]

\[ \text{constraint eqns:} \]

\[ \begin{align*}
0 & : \quad y = \frac{L}{4} (\sin \alpha + \sqrt{3} \cos \alpha) \quad \Rightarrow \quad S_y = \frac{L}{4} (\cos \alpha - \sqrt{3} \sin \alpha) S \alpha \\
0 & : \quad x = \frac{L}{4} (\cos \alpha - \sqrt{3} \sin \alpha) \quad \Rightarrow \quad S_x = -\frac{L}{4} (\sin \alpha + \sqrt{3} \cos \alpha) S \alpha \\
\end{align*} \]

\[ F_x = -mg \quad \Rightarrow \quad F_y = -mg \quad ; \quad F_x = 0 \quad \Rightarrow \quad F_y = mg \]

Virtual work:

\[ SW = -mg S y = -\frac{mgL}{4} (\cos \alpha - \sqrt{3} \sin \alpha) S \alpha \]

Principle of virtual work:

At equilibrium

\[ SW = 0 \text{ for arbitrary } S \alpha \]

or

\[ Q_x = 0 \]

\[ \cos \alpha - \sqrt{3} \sin \alpha = 0 \quad \Rightarrow \quad \tan \alpha = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = 30^\circ \]

Lagrange multiplier method:

Constraint eqn \( 0 \):

\[ \frac{\partial F_x}{\partial x} + \lambda_1 \frac{\partial S_x}{\partial x} + \lambda_2 \frac{\partial S_y}{\partial x} = 0 \Rightarrow \lambda_2 = 0 \]

Constraint eqn \( 0 \):

\[ \frac{\partial F_y}{\partial y} + \lambda_1 \frac{\partial S_x}{\partial y} + \lambda_2 \frac{\partial S_y}{\partial y} = 0 \Rightarrow \lambda_1 = mg \]

\[ \lambda_2 = 0 \]

\[ \lambda_1 = mg \]

\[ \frac{\partial F_x}{\partial x} + \lambda_1 \frac{\partial S_x}{\partial x} + \lambda_2 \frac{\partial S_y}{\partial x} = 0 \Rightarrow \lambda_1 \frac{\sqrt{3} \sin \alpha - \cos \alpha}{\cos \alpha} + \lambda_2 \frac{\sqrt{3} \sin \alpha - \cos \alpha}{\cos \alpha} = 0 \]

\[ \Rightarrow \quad \lambda = 0 \quad \Rightarrow \quad \tan \alpha = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = 30^\circ \]
Example: static equilibrium using virtual work:

\[ y = -Cx^2 \]

Find \( x \) and \( y \) for bead at equilibrium.

**Method I:** Generalized coord. \( q = x \), \( y = -Cq^2 \)

Eliminate \( x \) and \( y \) in favor of simple coord. \( q \):

\[ SW = -k(q-L)S_q + mg2CqS_q = -k(q-L) + 2Cmgq \]

\[ Q(q) = 0 \]

Note that:

\[ Q = F_x \frac{\partial x}{\partial q} + F_y \frac{\partial y}{\partial q} \]

\[ Q = \frac{kl - kq + 2Cmgq}{(k - 2Cmg)} \]

**Method II:**

\[ F_x S_x + F_y S_y + \lambda S_q + 2Cmx S_x = 0 \]

or

\[ \left[ -k(x-L) + 2Cmx \right] S_x + (\lambda - mg)S_y = 0 \]

\[ \lambda = mg \]

\[ x = \frac{kl}{(k - 2Cmg)} \]

Note the form of constraint drop:

\[ F_x^{(0)} = \lambda t = mg \]

\[ F_y^{(0)} = +kLx \]

\[ x = \frac{kl}{(k - 2Cmg)} \]

as expected.
Obtaining Lagrange's Equations from \( F_i = \dot{P}_i \) by extending the principle of virtual work. (D'Alembert's Principle):

The principle of virtual work is only useful for finding the static equilibrium configuration.

Now we modify principle of virtual work to include dynamics.

For equilibrium we used

\[
F_i^{\text{total}} = 0
\]

to obtain principle of virtual work which eliminated the forces of constraint from the problem.

Including dynamics we have (from Newton's 2nd)

\[
F_i^{\text{total}} = \ddot{P}_i - \left( \ddot{P}_i = \frac{dP_i}{dt} = m_i \frac{d^2 \mathbf{r}_i}{dt^2} \right)
\]

Thus we consider

\[
F_i^{\text{total}} - \ddot{P}_i = 0 \quad \text{(at any time } t) \]

Now let system undergo a virtual displacement (infinitesimal displacement consistent with the constraints at time \( t \) which is fixed):

\[
\sum_i (F_i^{\text{total}} - \ddot{P}_i) \cdot \mathbf{s} \mathbf{r}_i = 0
\]

As before, \( F_i^{\text{total}} = F_i^{(a)} + F_i^{(c)} \) and we assume the forces of constraint do no work.
Then \[ \sum_{i=1}^{N} \left( \mathbf{F}_i^{(a)} - \mathbf{F}_i^{(b)} \right) \cdot \delta \mathbf{x}_i = 0 \]  

**D’Alembert’s Principle.**

(Drop the \( ^{(a)} \) from here on.)

Assume holonomic constraints: \( N \) particle constraints and \( m \) constraint equations.

Use generalized coordinates: \( q_1, q_2, ..., q_n \) \( n = q \frac{n}{2} \) \( (n = 3N - m) \)

Then, as before

\[ \delta \mathbf{x}_i = \mathbf{x}_i \left( \mathbf{q} \frac{n}{2}, t \right) \]

and virtual displacements:

\[ \delta \mathbf{x}_i = \sum_{k=1}^{n} \mathbf{q}_k \frac{n}{2} \delta q_k \]

D’Alembert’s Principle becomes:

\[ \sum_{i=1}^{N} \left( \mathbf{F}_i^{(b)} - m_i \ddot{q}_i \right) \cdot \delta \mathbf{x}_i = \sum_{i=1}^{N} \left( \mathbf{F}_i^{(b)} \cdot \delta \mathbf{x}_i - m_i \ddot{q}_i \cdot \delta \mathbf{x}_i \right) = 0 \]

\[ \left( \text{Note: } \delta \mathbf{x}_i \text{ defined, } \delta \mathbf{x}_i \text{ depend on } \mathbf{q} \frac{n}{2} \right) \]

Need to work on the second term: forces, \( \mathbf{F}_i^{(b)} \) depend on \( \mathbf{q} \frac{n}{2} \).

\[ m_i \ddot{q}_i = \frac{d}{dt} \left( m_i \dot{q}_i \cdot \delta \mathbf{x}_i \right) - m_i \dot{q}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{q}_i}{\partial q_k} \]
Note that \( \mathbf{r}_i = \mathbf{r}_i^0 (q_{i1}, \ldots, q_{in}, \dot{q}_i) \) and \( \dot{q}_i = \dot{q}_i(t) \), thus

\[
\dot{\mathbf{v}}_i = \dot{\mathbf{r}}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_{ik}} \dot{q}_{ik} + \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i} \dot{\dot{q}}_i
\]

and

\[
\frac{\partial \dot{\mathbf{v}}_i}{\partial \dot{q}_i} = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_{ik}} \dot{q}_{ik} + \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i} \dot{\dot{q}}_i = \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i}
\]

as \( \frac{\partial \dot{\mathbf{v}}_i}{\partial \dot{q}_i} \) when operating on \( \dot{\mathbf{v}}_i \) ("Cancellation of dots"

(\( \mathbf{r}_i \) is fn. of \( q \) is not \( \ddot{q}_i \)).

Thus we have

\[
-m_i \ddot{r}_i = -\frac{d}{dt} (m_i \dot{r}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i}) - m_i \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i} \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial \dot{q}_i}
\]

\[
= -\frac{d}{dt} (\frac{1}{2} m_i \dot{r}_i^2) - \frac{\partial}{\partial \dot{q}_i} (\frac{1}{2} m_i \dot{r}_i^2)
\]

Hence, d'Alembert's Principle becomes

\[
\sum_l \left[ Q_l - \frac{\partial}{\partial q_{il}} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_{il}} + \frac{\partial T}{\partial q_{il}} \right] \delta q_{il} = 0
\]

But the \( \delta q_{il} \)'s are independent and hence

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_{il}} - \frac{\partial T}{\partial q_{il}} = Q_l \quad \text{for all } l
\]

\( \text{Generalized force Lagrange Equations} \)

where \( T = \sum_{i} \frac{1}{2} m_i \dot{r}_i^2 \) \( \equiv \) Kinetic Energy of System.
The applied forces can always be split into two types:

\[ F^e = F^v + F^{NV} \]

where \( F^p \) are forces that can be expressed in terms of a time dependent potential function \( V(\{\mathbf{r}_i\}, t) \)

"Potential Forces" \( \equiv F^v = -\nabla_{\mathbf{r}_e} V(\{\mathbf{r}_i\}, t) \)

\[ V(\{\mathbf{r}_i\}, t) = \sum_i V_i(\mathbf{r}_i, t) + \frac{1}{2} \sum_{i \neq k} V_{ik}(\mathbf{r}_i, \mathbf{r}_k) \]

"Non-potential forces" \( \equiv F^{NP} \) are all remaining forces that cannot be expressed in terms of any potential function.

Then the generalized forces, \( \mathbf{Q}_e = \mathbf{Q}^p + \mathbf{Q}^{NP} \),

\[ \mathbf{Q}_e^v = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{F}_i}{\partial \mathbf{q}_e} = \sum_{i=1}^{N} -\nabla_{\mathbf{r}_i} V \cdot \frac{\partial \mathbf{F}_i}{\partial \mathbf{q}_e} = -\sum_{j=1}^{3N} \frac{\partial V}{\partial x_j} \frac{\partial x_j}{\partial q_e} = -\frac{\partial V}{\partial q_e} \]

\( \mathbf{Q}_e^{NP} \) are all remaining forces that cannot be expressed in terms of any potential function.

\[ \mathbf{Q}_e^{NP} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{E}_i}{\partial \mathbf{q}_e} = \mathbf{Q}_e^{NP}(\{\mathbf{q}_i\}, \{\mathbf{v}_i\}) \]

Then the generalized force form of Lagrange Eqs. become:

For each \( \dot{q}_l \):

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_l} \right) - \frac{\partial T}{\partial q_l} = \frac{\partial V}{\partial q_l} + \sum_l \mathbf{Q}^{NV}_l \]

or since \( V \) does not depend on \( \mathbf{q}_i \)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_l} \right) - \frac{\partial L}{\partial q_l} = \mathbf{Q}_l^{NV} \]

where \( L = T - V \equiv \) The Lagrangian for the system

If non-potential forces are zero, i.e. \( \mathbf{Q}^{NP}_l = 0 \) for all \( l \)

then

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_l} \right) - \frac{\partial L}{\partial q_l} = 0 \] for each \( l \) (Lagrange Eqs.)
First set of Examples using Lagrangian Formalism.

Example 1:

Use Lagrangian method to find the motion of each mass.

Choose generalized coord. $(x, s)$

\[ T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2) \quad x_m = x + s \cos \alpha \]
\[ y_m = s \sin \alpha \]

\[ T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[ (x + \dot{x} \cos \alpha)^2 + \dot{s}^2 \sin^2 \alpha \right] \]

\[ T(x, \dot{s}) = \frac{1}{2} (M+m) \dot{x}^2 + m \dot{x} \dot{s} \cos \alpha + \frac{1}{2} m \dot{s}^2 \]

\[ V = m g \dot{y}_m = m g s \sin \alpha \]

Thus,

\[ L(x, \dot{x}, \dot{s}, t) = T - V = \frac{1}{2} (M+m) \dot{x}^2 + m \cos \alpha \dot{x} \dot{s} + \frac{1}{2} m \dot{s}^2 - m g s \sin \alpha \]

\[ \frac{\partial L}{\partial x} = (M+m) \ddot{x} + m \cos \alpha \dot{s} \quad \frac{\partial L}{\partial \dot{x}} = 0 \]

\[ \frac{\partial L}{\partial s} = m \ddot{s} + m \cos \alpha \dot{x} \quad \frac{\partial L}{\partial \dot{s}} = -m g \sin \alpha \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow (M+m) \ddot{x} + m \cos \alpha \ddot{s} = 0 \quad (1) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0 \Rightarrow m \dddot{s} + m \cos \alpha \dot{x} + mg \sin \alpha = 0 \quad (2) \]

From (1) $\dot{x} = -\frac{m \cos \alpha}{M+m} \ddot{s}$. Put this into (2):

\[ \left( 1 - \frac{m \cos \alpha}{M+m} \right) \ddot{s} = -g \sin \alpha \]

or

\[ \ddot{s} = -g \frac{\sin \alpha}{\left[ 1 - \frac{m \cos \alpha}{M+m} \right]} \]

Note if $M \rightarrow \infty$ then have stationary incline plane problem.

\[ M \rightarrow \infty \Rightarrow \ddot{s} = -g \frac{\sin \alpha}{\sin \alpha} \]

If $M=0 \Rightarrow \ddot{s} = -\frac{g}{\sin \alpha}$

or $\ddot{s} \sin \alpha = -g$ as expected.
Example 2: Moving constraint example: Incline plane accel. to right with acceleration $a$.

(In the previous problem the incline plane moves in a way which must be determined by solving the problem. Now we suppose we are given how the plane moves at the outset of the problem.)

Given: Constraint equation $x(t) = \frac{1}{2} a t^2$.

Choose generalized coord. $s$ (x is now known because of the constraint):

\[ x_m = x(t) + s \cos \alpha \]
\[ x_m = x + \frac{1}{2} a t^2 \]
\[ x = x_0 + v_0 t + \frac{1}{2} a t^2 \]

For simplicity take $x_0 = v_0 = 0$.

\[ x = \frac{1}{2} a t^2 \]
\[ \dot{x} = at \]

Expression for $T$ and $V$ are the same as for Example 1 but $x$ and $\dot{x}$ are now known:

\[ T(s, \dot{s}, t) = \frac{1}{2} m \dot{s}^2 + m a t \cos \alpha \dot{s} + \frac{1}{2} m s^2 \]
\[ V = m g s \sin \alpha \]

\[ L(s, \dot{s}, \dot{t}) = T - V \]
\[ \frac{\partial}{\partial s} L = m a \cos \alpha + m \dot{s} \]
\[ \frac{\partial}{\partial \dot{s}} L = -m g \sin \alpha \]

\[ \frac{\partial}{\partial \dot{s}} L - \frac{\partial}{\partial \dot{s}} V = 0 \Rightarrow m a \cos \alpha + m \dot{s} + m g \sin \alpha = 0 \]

or

\[ \ddot{s} = -g \sin \alpha - a \cos \alpha \]

Check: Find $a$ for which $\ddot{s} = 0$:

\[ a = -g \tan \alpha \]

\[ N \sin \alpha = m a \]
\[ N \cos \alpha = m g \]

\[ a = g \sin \alpha \cos \alpha \]

\[ a = g \sin \alpha \cos \alpha \]

and $N = mg / \cos \alpha$
Example 3

\[ A t \ t = 0 : \ x = l, \ \phi = 0, \ y = 0. \]

Constraint:
\[ y = l - x + R \dot{\phi} \]
\[ \therefore \dot{y} = R \ddot{\phi} - \ddot{x} \]

Take as generalized coord.: \( (x, \ \phi) \)

\[
T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{\phi}^2
\]
\[
= \frac{1}{2} M \dot{x}^2 + \frac{1}{4} m R^2 \dot{\phi}^2 + \frac{1}{2} m (R \ddot{\phi} - \dot{x})^2
\]
\[
T = \frac{1}{2} (M + m) \dot{x}^2 - m R \dot{x} \dot{\phi} + \frac{1}{2} (m + \frac{1}{2} m) R^2 \dot{\phi}^2
\]

and
\[ V = -m g y = -m g (l - x + R \dot{\phi}) \]

\[ L = T - V, \ \frac{\partial L}{\partial \dot{x}} = (M + m) \ddot{x} - m R \ddot{\phi} ; \ \frac{\partial L}{\partial \dot{\phi}} = -m g \]

\[ \frac{\partial L}{\partial \phi} = (m + \frac{1}{2} m) R^2 \dddot{\phi} - m R \ddot{x} \ ; \ \frac{\partial L}{\partial \phi} = m g R \]

\[ \therefore \text{Lagrange's equations:} \]
\[ x: \quad (M + m) \dddot{x} - m R \dddot{\phi} + m g = 0 \quad \text{(1)} \]
\[ \phi: \quad (m + \frac{1}{2} m) R^2 \dddot{\phi} - m R \dddot{x} - m g R = 0 \quad \text{(2)} \]

From (1) \[ R^2 \dddot{\phi} = \frac{(M + m) \dddot{x}}{m} + g R \] ; substitute into (2)

\[
\left[ (m + \frac{1}{2} m) \frac{(M + m)}{m} R - m R \right] \dddot{x} + m g R + \frac{1}{2} M g R - m g R = 0
\]

\[
\left[ \frac{3}{2} M + \frac{1}{2} m^2 \right] \dddot{x} = \frac{-1}{2} M g R
\]

\[ \dddot{x} = -\frac{m g}{[M + 3m]} \]

\[ \dddot{x} = -\frac{m g}{[M + 3m]} \quad \text{\( x \equiv a_{cm} \text{ in sol'n} \)} \]
Special velocity dependent forces:

I. Generalized Potential - Applicable to Electro-magnetic forces.

II. Frictional forces - Rayleigh's Dissipation Function.

I. Generalized Potential:

For usual potential function: \( V = V(\frac{\partial x}{\partial z}, t) \); \( \mathbf{F} = -\nabla V \)

or (see p. 2-18)

\[ Q_j(\frac{\partial x}{\partial z}, t) = -\frac{\partial V}{\partial q_j} \]

where \( V = V(\frac{\partial x}{\partial z}, t) \). Note \( V \) does not depend on \( q_j \)’s.

Thus, velocity dependent forces (such as electro-magnetic forces on charged pole) cannot be expressed in terms of the usual potential function. These velo dep. forces had to be included in \( Q^{NW} \).

However, if we can find a function \( U = U(\frac{\partial x}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial x}{\partial z}) \) such that

\[ Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial q_j}\right) \]

then \( L = T - U \) satisfies

\[ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j. \]

This follows immediately from \( \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \).

Where \( Q_j \) are remaining forces that cannot be expressed in terms of gen. pot. in.

\( U \) is called Generalized Potential.

Note if the generalized coord. are the standard cartesian coord.

\[ \frac{\partial x}{\partial z} = \frac{\partial x}{\partial z} \quad \text{and} \quad \frac{\partial x}{\partial z} = \frac{\partial x}{\partial z} = \frac{\partial x}{\partial z} = \frac{\partial x}{\partial z} \quad \text{and} \quad Q_j \to F_j \]

\[ F_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \quad U(\{\dot{q}_j\}, \{q_j\}, t). \]
An example of a velocity-dependent force which can be expressed in terms of a generalized potential is electromagnetic force.

\[ F = q \left\{ \mathbf{E} + \mathbf{v} \times \mathbf{B} \right\} \quad \text{in MKS units}, \]

or

\[ F = q \left\{ -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right\} \]

where \( \phi(\mathbf{r},t) \) and \( \mathbf{A}(\mathbf{r},t) \) are the usual scalar and vector potential.

\( \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \).

Some vector analysis gives

\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla) \mathbf{A} \]

and hence

\[ \begin{align*}
F &= q \left\{ -\nabla \phi (\mathbf{r},\mathbf{v},t) - \frac{d \mathbf{A}}{dt} \right\} \\

\text{generalized potential} & \quad U(\mathbf{r},\mathbf{v},t) \quad \text{such that} \\
F &= -\nabla U + \frac{d}{dt} (\nabla U) \\

\text{Note} \quad \frac{d \mathbf{A}}{dt} = \frac{d}{dt} (\nabla (\nabla \cdot \mathbf{A})) \quad \text{since} \quad \mathbf{A} \text{ does not depend on} \mathbf{v} \\
\nabla (\nabla \cdot \mathbf{A}) &= (\nabla \cdot \nabla) \mathbf{A} + \mathbf{A} \\

\therefore \quad \text{The generalized potential function we are looking for is} \\
U(\mathbf{r},\mathbf{v},t) &= q \phi(\mathbf{r},t) - q \mathbf{v} \cdot \mathbf{A}(\mathbf{r},t) \\
\text{and thus the Lagrangian is} \\
L = T - U = T - q \phi + q \mathbf{v} \cdot \mathbf{A} \]
II. Velocity dependent friction:

Suppose \[ F_{ti} = -k_x \dot{X}_i - k_y \dot{Y}_i - k_z \dot{Z}_i \]

Define Rayleigh dissipation function:

\[ \dot{f}(\nu, \dot{\nu}) = \frac{1}{2} \sum_i (k_x \nu_i \dot{X}_i + k_y \nu_i \dot{Y}_i + k_z \nu_i \dot{Z}_i) \]

Then

\[ F_{ti} = -\nabla_{\nu_i} f \]

In terms of generalized cords:

\[ Q_{il} = \sum_i F_{ti} \cdot \frac{\partial \nu_i}{\partial q_l} = \sum_i F_{ti} \cdot \frac{\partial \nu_i}{\partial q_l} = \sum_i \left( \dot{\nu}_i \frac{\partial f}{\partial q_l} \right) = -\frac{\partial f}{\partial q_l} \]

Now in general, can split forces into those expressible in terms of a generalized potential and those which cannot. Leave generalize forces \( Q^\text{lin} \) for those forces which cannot be expressible in terms of generalized potential. Then have

\[ \frac{d}{dt} \frac{\partial f}{\partial q_l} - \frac{\partial f}{\partial q_l} = Q_{il} \]

In the case \( Q_{il} = Q_f i \) and we have

\[ \frac{d}{dt} \frac{\partial f}{\partial q_l} - \frac{\partial f}{\partial q_l} + \frac{\partial f}{\partial q_l} = 0. \]
On bringing constraint forces into the problem (using D'Alambert's Principle)

Suppose we have a system with general coordinates \( q = (q_1, \ldots, q_n) \).

Suppose all forces (except constraint forces) are given by a general potential, \( U(q, \dot{q}, t) \).

Then D'Alambert's Principle can be written

\[
\left( \frac{\partial \mu}{\partial t} - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_l} \right) - \frac{\partial U}{\partial q_l} \right) \delta q_l = 0 \tag{sum rule}
\]

But

\[
\frac{\partial \mu}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial \dot{q}_l} \right) \frac{\partial T}{\partial \dot{q}_l} - \frac{\partial U}{\partial q_l}
\]

\[Q^\mu_l = 0 \quad \text{for all } l
\]

So we have

\[
\left( \frac{\partial}{\partial t} \left( \frac{\partial}{\partial \dot{q}_l} \right) - \frac{\partial U}{\partial q_l} \right) \delta q_l = 0 \quad \text{for arbitrary } \delta q_l
\]

This leads to

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{q}_l} \frac{\partial T}{\partial \dot{q}_l} = 0 \quad \text{for each } l
\]

Now suppose we add a new constraint \( G(q, t) = 0 \).

Then, if we let a virtual displacement consistent with the constraints (including the new one) occur, we have

\[
DG = \frac{\partial G}{\partial \dot{q}_l} \delta q_l + \frac{\partial G}{\partial t} \delta t = 0
\]

and for a virtual displacement \( \delta t = 0 \) and

\[
\delta G = \frac{\partial G}{\partial \dot{q}_l} \delta q_l = 0
\]

shows that the \( \delta G \)'s are independent of \( \delta q_l \) and the \( \delta G \)'s are independent of \( \delta t \).

Now, if we could add a force, say, \( Q^\mu_l = Q^\mu_l \) that is equal to the constraint force, then we could treat the \( \delta G \)'s as independent, because the \( Q^\mu_l \) would
For the system to move consistent with the added constraint,

**Lagrange Multiplier Method:**

We do not know what the constraint force is, but we do know that it should do no virtual work.

\[ SG = \frac{\partial G}{\partial q_k} = 0 \]

and

\[ \lambda \frac{\partial G}{\partial q_k} = \frac{\partial G}{\partial q_k} \frac{\partial q_k}{\partial \lambda} = 0 \quad \lambda \text{ is not known} \]

\[ \lambda(t) \] is called a Lagrange multiplier. For \( \lambda \) can be different at a different \( t \).

So \( \lambda = \lambda(t) \)

Suppose we put \( \frac{\partial G}{\partial q_k} = \lambda \frac{\partial q_k}{\partial \lambda} \), then \( G^c \frac{\partial q_k}{\partial \lambda} = 0 \)

as is required for the constraint force.

Then we have

\[ \left( \lambda \frac{\partial G}{\partial q_k} - \left[ \frac{d}{dt} \frac{\partial G}{\partial \dot{q}_k} - \frac{\partial G}{\partial q_k} \right] \right) \frac{\partial q_k}{\partial \lambda} = 0 \text{ for arbitrary } \dot{q}_k. \]

\[ \left( \lambda \frac{\partial G}{\partial q_k} - \frac{d}{dt} \frac{\partial G}{\partial \dot{q}_k} - \frac{\partial G}{\partial q_k} \right) \frac{\partial q_k}{\partial \lambda} = 0 \text{ for each } \dot{q}_k. \]

or

\[ \frac{d}{dt} \left( \lambda \frac{\partial G}{\partial q_k} \right) - \lambda \frac{d}{dt} \frac{\partial G}{\partial \dot{q}_k} = \lambda \frac{\partial G}{\partial q_k} \]

Together with the constraint eq.

Either \( G(q,t) = 0 \) or \( \frac{\partial G}{\partial q_k} \frac{\partial q_k}{\partial \lambda} + \frac{\partial G}{\partial q_k} = 0 \) (1 eq.)

This gives \( n+1 \) eqs. to solve for the \( n \) generalized coordinates \( q_1(t), q_2(t), \ldots, q_n(t) \) and the Lagrange multiplier function \( \lambda(t) \).

After solving for \( \text{eq}(t) \) and \( \lambda(t) \), the constraint forces are then

\[ q^c_k = \lambda \frac{\partial G}{\partial q_k}. \]
Non-holonomic constraints:

I consider here only non-holonomic differential constraints that lead to constraint eqs. that are linear in the velocities.

Differential constraints of the form

$$a(q, \dot{q}, t) \dot{q} \cdot q + a_t(q, \dot{q}, t) \dot{q} = 0 \quad \text{(sum rule)}$$

lead to linear velocity constraints, e.g.,

$$\dot{q}^i(q, \dot{q}, t) = a(q, \dot{q}, t) \ddot{q}^i + a_t(q, \dot{q}, t) = 0$$

(The important restriction is that the functions $a^i$ and $a_t$ do not depend on $\dot{q}$)

Note: If the differential constraint is non-holonomic, then there is no function $G(q, t)$ such that

$$dG = \dot{a}(q, \dot{q}, t) \frac{\partial G}{\partial \dot{q}} + a_t(q, \dot{q}, t) \frac{\partial G}{\partial t}$$

i.e., there is no fn. $G(q, t)$ s.t. $a^i = \frac{\partial G}{\partial \dot{q}^i}$ and $a_t = \frac{\partial G}{\partial t}$.

And there is no integrating factor that will make such a function $G$ possible.

This class of non-holonomic constraints can be treated in the same way as holonomic constraints with

$$\frac{\partial G}{\partial \dot{q}} \rightarrow a(q, \dot{q}, t) = \frac{\partial G}{\partial \dot{q}}$$

and

$$\frac{\partial G}{\partial t} \rightarrow a_t(q, \dot{q}, t) = \frac{\partial G}{\partial t}$$

$$Q_t^c = \lambda \frac{\partial G}{\partial \dot{q}} \rightarrow \lambda a(q, \dot{q}, t) = \lambda \frac{\partial G}{\partial \dot{q}}$$
The Lagrange Multiplier method is easily extended to the constraint force arising from several constraints. (All holonomic, all non-holonomic, or a mixture.)

Suppose we have \( n_c \) constraints for which we want to find the constraint forces.

We assume the differential constraints are of the form

\[
\alpha_{x\ell}(x,t)\frac{dx_{\ell}}{dt} + \alpha_{x\ell}(x,t) dt = 0 \quad \text{for } \alpha = 1, \ldots, n_c.
\]

If \( \alpha \)-th constraint is holonomic, \( G_{\alpha}(x,t) \), then

\[
\alpha_{x\ell} = \frac{\partial G_{\alpha}}{\partial x_{\ell}}.
\]

Then introduce \( n_c \) Lagrange multiplier funs. \( \lambda_{\alpha}(t) \)

The Lagrange Eqs. with gen constraint forces are:

\[
\begin{align*}
(n_c \text{ egs}) \Rightarrow \quad \lambda_t \frac{\partial L}{\partial \dot{x}_{\ell}} - \frac{\partial L}{\partial x_{\ell}} &= \sum_{\alpha} \lambda_{\alpha} \alpha_{x\ell} \\
\end{align*}
\]

Together with the velocity constraint egs

\[
\begin{align*}
(n_c \text{ egs}) \Rightarrow \quad \alpha_{x\ell} \dot{x}_{\ell} + \alpha_{\ell t} &= 0 \quad \text{for } G_{\alpha}(x,t) = 0 \text{ if a holonomic constraint}.
\end{align*}
\]

This gives \( n + n_c \) egs. for the \( n \) gen coords, \( \dot{x}_{\ell}(t) \) and the \( n_c \) Lagrange multiplier funs. \( \lambda_{\alpha}(t) \).

Having solved for \( \{\dot{x}_{\ell}(t)\}^T \) and \( \{\lambda(t)\}^T \), the constraint forces are

\[
\dot{Q}_{\ell}^c = \sum_{\alpha} \dot{Q}_{x\ell}^c \quad \text{where } Q_{x\ell}^c = \lambda_{\alpha}(t) \alpha_{x\ell}(x(t),t)
\]
Additional cautionary note:

There are often attempts to extend these methods to more general velocity constraint equations, such as

\[ \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = 0 \]

where \( \dot{\mathbf{q}} \)'s do not appear as linear forms in \( \mathbf{q} \).

There seem to be many pitfalls in these methods and they should be used with very great care.

For example, our text, Goldstein, Poole, & Safko 3rd ed, had a completely erroneous section in chapter 2 on this matter that persisted up until the 6-th printing — when it was removed in 2004.

There are several references in addition to the two given at the bottom of pg 47 in GPS. A good recent reference seems to be: