Consider rigid body consisting of $N$ particles.

How many coordinates are required to completely specify the location and orientation of the rigid body?

$$n = 3N - \left( \# \text{ of independent constraint equations} \right)$$

The constraint equations for a rigid body are of the form

$$r_{ij} = \text{const}.$$ 

There are $\frac{1}{2}N(N-1)$ of these equations and if they were all independent there would have

$$3N - \frac{N^2}{2} + \frac{N}{2} = N(3 - \frac{N}{2} + \frac{1}{4}) = N(7 - N) < 0 \quad \text{if} \quad N > 7.$$

Obviously, all the constraint equations are not independent.

Number of independent constraint equations for rigid body:

Pick three index points 1, 2, and 3 (not collinear)

These three points are rigidly fixed by three constraints: $c_{12}, c_{13}, c_{23}.$

And each of the other particles is fixed by giving the distances to these three.

Need $3 + 3(N-3) = 3N - 6$ constraint equations.

Thus, the number of independent equations:

$$n = 3N - (3N - 6) = 6$$

for location of a ref. point, usually the C.M. 

3 for orientation
To specify the orientation of the rigid body we embed a coord. system in the body with origin at C.M.

We specify the orientation of the rigid body by giving the rotations that must be carried out to get from the unprimed coord. to the primed.

First, we will review rotations in terms of orthogonal transformations.

Consider a vector \( \mathbf{C} \) represented in two coord. systems with coincident origins.

\[
\mathbf{C} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3
\]

\( \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \) \equiv \text{unit vectors in } x, y, z \text{ direction}

\[
\mathbf{C}' = c_1' \mathbf{X}_1' + c_2' \mathbf{X}_2' + c_3' \mathbf{X}_3'
\]

Now the \( c'_i \)'s are linearly related to the \( c_i \)'s:

\[
\begin{align*}
\mathbf{X}_1' & = a_{11} \mathbf{X}_1 + a_{12} \mathbf{X}_2 + a_{13} \mathbf{X}_3 \\
\mathbf{X}_2' & = a_{21} \mathbf{X}_1 + a_{22} \mathbf{X}_2 + a_{23} \mathbf{X}_3 \\
\mathbf{X}_3' & = a_{31} \mathbf{X}_1 + a_{32} \mathbf{X}_2 + a_{33} \mathbf{X}_3
\end{align*}
\]

Note: \( a_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j \). etc.

Thus \( a_{ij} = (\mathbf{X}_i \cdot \mathbf{X}_j) \).

Since it represents the same vector, clearly the length of \( \mathbf{C} = \text{length of } \mathbf{C}' \).
For a "rotation" the length of $c$ must be unchanged:

Thus $c \cdot c = c' \cdot c'$

$$c \cdot c = c_1 c_1 + c_2 c_2 + c_3 c_3 = c'_1 c'_1 + c'_2 c'_2 + c'_3 c'_3$$

**Notation:**

$$\frac{2}{5} c_{\mu} c_{\mu} = c_{2} c_{2}$$  \hspace{1cm} \text{(automatic summation over repeated index,)}

\hspace{1cm} \text{(summation convention first introduced by Einstein)}

Thus $c \cdot c = c_\mu c_\nu = c'_\mu c'_\nu$

Also $c'_\mu = \delta_{\mu\nu} c_\nu$

\therefore $c'_\mu c'_\mu = \delta_{\mu\nu} c_\mu c_\nu = \delta_{\mu\nu} c_\mu c_\nu$  \hspace{1cm} \text{(The Kronecker delta)}

\therefore $c_2 c_2 = \delta_{\mu\nu} c_\mu c_\nu$  \hspace{1cm} \text{(for any vector $c$)}

This requires $\delta_{\mu\nu} \delta_{\mu\nu} = \delta_{\nu\nu}$

\hspace{1cm} \text{where $\delta_{\nu\nu} = \begin{cases} 1 & \text{if } \nu = \nu' \\ 0 & \text{otherwise} \end{cases}$

$\nu, \nu' = 1, 2, 3$.

This property of the transformation matrix of $a_{\mu\nu}$'s is called the "orthogonal" transformation — in particular such a transformation leaves lengths unchanged.

Note we could write $a_{\mu\nu}$ and $c$ and $c'$ in matrix form:

$$c' = \tilde{A} c$$

where $\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}; \quad c' = \begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix}$$

**Orthogonal matrix satisfies the condition**

\begin{align*}
\delta_{\mu\nu} a_{\mu\nu} &= \delta_{\nu\nu} \\
\mu &\text{ is summed over } 1, 2, 3
\end{align*}
Consider a rotation about a single axis; (say $x_3$)

$$C = c_1 \hat{x}_1 + c_2 \hat{x}_2 + c_3 \hat{x}_3$$

$$C' = c_1 \hat{x}_1' + c_2 \hat{x}_2' + c_3 \hat{x}_3'$$

How are the components $c_1'$ and $c_2'$ related to the components $c_1$ and $c_2$.

From the geometry shown above

$$c_1' = c_1 \cos \theta + c_2 \sin \theta$$

$$c_2' = -c_1 \sin \theta + c_2 \cos \theta$$

($c_3' = c_3$)

$$C' = \hat{A} C$$

where

$$\hat{A} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Note:**

$$\alpha_{12} = \hat{x}_i \cdot \hat{x}_j$$

$$\alpha_{12} = \cos \phi = \cos(\theta - \phi) = \cos \theta$$

$$\alpha_{21} = \hat{x}_1' \cdot \hat{x}_1$$

$$\alpha_{21} = \sin(\alpha - \beta) = -\sin \theta$$

Active vs Passive

However, this could equally well be viewed as keeping the coord. system fixed and rotating the vector $C$ through an angle $\theta$.

$$C = c_1 \hat{x}_1 + c_2 \hat{x}_2$$

$$C' = c_1 \cos(\phi - \theta) \hat{x}_1 + c_2 \sin(\phi - \theta) \hat{x}_2$$

$$= (c_1 \cos \phi \cos \theta + c_2 \sin \phi \sin \theta) \hat{x}_1$$

$$+ (c_2 \cos \phi \sin \theta - c_1 \sin \phi \cos \theta) \hat{x}_2$$

$$\therefore c_1' = \cos \theta \ c_1 + \sin \theta \ c_2$$

and $c_2' = -\sin \theta \ c_1 + \cos \theta \ c_2$

or $C' = \hat{A} C$ as promised.
In general, $C' = \hat{A} C$ can be considered in two different ways:

(a) passive: $C'$ represents the same vector $C$ viewed from a new coord. system $x'y'z'$ rotated by $\hat{A}$

(b) active: $C'$ represents a new vector rotated by $\hat{A}$ viewed from the same coord. system.

Of course, if $\hat{A}$ viewed in the passive sense rotates the coord. system counterclockwise about some axis, then $\hat{A}$ viewed in the active sense rotates the vector $C$ counterclockwise about the same axis.

Review of the formal properties of orthogonal transformations.

1.) Two successive finite rotations: $\hat{A}$ then $\hat{B}$

$R' = \hat{A} R$

$R'' = \hat{B} (\hat{A} R)$

$R'' = \hat{C} R$

$C_{\mu} = b_{\alpha\\mu} a_{\alpha\\nu} R_{\nu}$

$C_{\mu} = b_{\alpha\\mu} a_{\alpha\\nu} R_{\nu}$

$\therefore \hat{C} = \hat{B} \hat{A}$ matrix multiplication.

Note that $\hat{B} \hat{A} \neq \hat{A} \hat{B}$ since in general $b_{\alpha\\mu} a_{\alpha\\nu} \neq a_{\alpha\\mu} b_{\alpha\\nu}$

Demonstration:

First rotate book about $y$ by $90^\circ$ then about $x$ by $90^\circ$.

First rotate book about $x$ by $90^\circ$ then about $y$ by $90^\circ$.

Thus, in general finite rotations do not commute.
Note that the successive finite rotations about same axis do commute.

For example: Rotations about $z$-axis:

$$\tilde{A}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{A}(\theta) \cdot \tilde{A}(\phi) = \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) & 0 \\ -\sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tilde{A}(\phi) \tilde{A}(\theta),$$

2. Inverse transformation: $\tilde{A}^{-1} \tilde{A} = \tilde{I}$ i.e. $\tilde{A}^{-1}$ puts vector back.

Also $(\tilde{A}^{-1})^T \tilde{A} = \tilde{I}$, i.e. $(\tilde{A}^{-1})^T = \tilde{A}^T$.

Proof: $(\tilde{A}^{-1})^T \tilde{A}^{-1} \tilde{A} = \tilde{I} \tilde{A}$

$$\tilde{I} \tilde{A} = \tilde{A} = \tilde{A} \tilde{I}.$$

Note: $(\tilde{A}^{-1}) = \frac{1}{\det \tilde{A}} \cdot \text{adj} \tilde{A},$ where $\text{adj} \tilde{A}$ is the adjugate matrix.

3. Note that the orthogonality condition is

$$\alpha_{\mu \nu} \alpha_{\mu \nu} = \delta_{\mu \nu},$$

$$\tilde{A}^T \alpha_{\mu \nu} \tilde{A} = \delta_{\mu \nu},$$

where $(\tilde{A}^T)_{\mu \phi} = \alpha_{\phi \mu}$.

Alternate statement of orthogonality condition:

$$\tilde{A}^T \tilde{A} = \tilde{I}$$

where $\tilde{A}^T$ is the transpose of $\tilde{A}$.

Note that this checks with relation about $z$-axis $\tilde{A}(\theta)$ above since $\tilde{A}(\theta)^{-1}$ is clearly equal to $\tilde{A}(-\theta) = \tilde{A}^T$ because $\sin(-\theta) = -\sin(\theta)$. 

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4. For orthogonal matrix, \( \det \hat{A} = |\hat{A}| = \pm 1 \).

Proof: \( |\hat{I}| = |\hat{A} \hat{A}^{-1}| = |\hat{A}| |\hat{A}^{-1}| \) (since \( \det A \cdot B = \det A \cdot \det B \))

but \( \hat{A}^{-1} = \hat{A}^t \) from property 3. above
and \( \det \hat{A}^t = \det \hat{A} \) since determinant is unchanged if interchange rows and columns

\[ |\hat{A}|^2 = 1 \Rightarrow |\hat{A}| = \pm 1, \quad (\text{if orthogonal matrix}) \]

Note \( |\hat{A}| = \pm 1 \) is necessary but not sufficient for orthogonal transformation. (Example \( \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) \( \det \hat{A} = -1 \)

but \( \hat{A}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \hat{A}^{-1} \) hence not orthogonal.

In this case \( \hat{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

\( \Rightarrow (\text{Not all orthogonal transformations represent rotations of rigid body}) \)

5. The orthogonal transformations which represent rotations of a rigid body has determinant \( = 1 \).

i.e. \( \det \hat{A} = |\hat{A}| = 1 \Rightarrow \text{proper rotation} \)

\( |\hat{A}| = -1 \Rightarrow \text{improper rotation} \)

\( \hat{A}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) has \( \det \hat{A}_1 = -1 \) and \( x' = -x \), \( y' = -y \), \( z' = z \)

Note \( \hat{A}_1^{-1} = \hat{A}_1 = \hat{A}_1^t \)

\( \Rightarrow \) \( \hat{C} \) has \( |\hat{C}| = -1 \)

then \( \hat{C} = \hat{A}_1 \cdot \hat{B} \Rightarrow \hat{E} = \hat{A}_1^{-1} \hat{C} \) and \( |\hat{C}| = |\hat{A}_1| |\hat{B}| = -1 \)

\( -1 \cdot |\hat{B}| = -1 \)

\( \Rightarrow \) any \( \hat{C} \) with \( |\hat{C}| = -1 \)

can be constructed from \( \hat{A}_1 \cdot \hat{B} \) = inversion rotation

\( \det \hat{B} = +1 \) \( \Rightarrow \text{impossible for rigid body} \).
6. Consider \( \tilde{A} \) transforms vector \( X \) to \( Y = \tilde{A}X \).

Now view this entire process from a primed co-ordinate system related to the original by transformation \( \tilde{B} \), i.e. \( X' = \tilde{B}X \) and \( Y' = \tilde{B}Y \).

What is the transformation \( \tilde{A}' \) in the primed co-ordinate system which transforms \( X' \) into \( Y' \), i.e., \( Y' = \tilde{A}'X' \)?

\[
\begin{align*}
&x(\tilde{B}x) \\
&\tilde{x} = \tilde{B}x \\
&y = \tilde{A}x \\
&y' = \tilde{A}'x' \\
&y = \tilde{B}y = \tilde{B}\tilde{A}x = \tilde{B}\tilde{A}\tilde{B}^{-1}x
\end{align*}
\]

Thus, \( \tilde{A}' = \tilde{B}\tilde{A}\tilde{B}^{-1} \) called a similarity transformation.

7. Extension of property 3 to complex matrices:

Define \( A^{\dagger} \) Adjoint matrix \( \tilde{A}^{\dagger} = (\tilde{A}^*)^* \) complex conjugate, transposed.

If \( \tilde{A}^{\dagger} \tilde{A} = \tilde{I} \) then \( \tilde{A} \) is unitary.

If \( \tilde{A}^{\dagger} = \tilde{A} \) then \( \tilde{A} \) is self-adjoint or hermitean.
Thus the orientation of a rigid body can be expressed at any time \( t \) by a transformation matrix \( \mathbf{A}(t) \).

\( \mathbf{A}(t) \) has 9 elements \( A_{ij} \) but they are not all independent because of the orthogonality conditions

\[ A_{\mu \nu} A_{\mu \nu} = \delta_{\nu \nu} \quad \text{(automatic condition)} \]

six different equations. (since \( A_{\mu \nu} A_{\mu \nu} = A_{\mu \nu} A_{\mu \nu} \))

\[
\begin{align*}
(\tau=1, \tau=1) & \Rightarrow A_{11} A_{11} = 1 \quad \text{and} \quad A_{12} A_{12} = 0 \quad \text{or} \quad (\tau=1, \tau=2) \\
(\tau=2, \tau=2) & \Rightarrow A_{22} A_{22} = 1 \quad \text{and} \quad A_{23} A_{23} = 0 \quad \text{or} \quad (\tau=2, \tau=3) \\
(\tau=3, \tau=3) & \Rightarrow A_{33} A_{33} = 1 \quad \text{and} \quad A_{31} A_{31} = 0 \quad \text{or} \quad (\tau=3, \tau=1)
\end{align*}
\]

This leaves \( 9 - 6 = 3 \) independent parameters to determine the orientation of the rigid body as expected. (bottom of pg 5-1)

One way to choose these three parameters is to use Euler Angles.
Definition of Euler Angles: (Successive rotations about axes fixed in the body)

1. Rotate about \( \zeta \) by angle \( \phi \)
\[ \mathbf{R} = \mathcal{D}_{\phi} \]

2. Rotate about \( \chi \) by angle \( \Theta \) (x-convention)
\[ \mathbf{R} = \mathcal{C}_{\Theta} \]

3. Rotate about \( \zeta'' \) by angle \( \psi \)
\[ \mathbf{R} = \mathcal{B}_{\psi} \]

\[ \mathbf{Z}(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \mathbf{X}''(\Theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta \sin \Theta & -\sin \Theta \cos \Theta \\ 0 & \sin \Theta \cos \Theta & \cos \Theta \end{pmatrix} \]

\[ \mathbf{Z}''(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \mathbf{R}' = D_{\phi} R' \]

\[ \mathbf{R}'' = C_{\Theta} R' \]

\[ \mathbf{R}''' = B_{\psi} R'' \]

\[ \mathbf{R}''' = A \mathbf{R} \]

Matrix multiplication gives

\[ \hat{A}_{\phi,\Theta,\psi} = \begin{pmatrix} \cos \psi \cos \phi & -\cos \Theta \sin \phi \sin \psi & \cos \psi \sin \phi + \cos \Theta \cos \phi \sin \psi \\ -\sin \psi \cos \phi & \cos \Theta \sin \phi \sin \psi & -\sin \psi \sin \phi + \cos \Theta \cos \phi \cos \psi \\ \sin \Theta \sin \phi & -\sin \Theta \cos \phi & \cos \Theta \end{pmatrix} \]
Note:
Other conventions for the choice of which axes and the order of their choice are possible.

For example: Engineers: $\mathbf{z}$ (given)
1. Rotation about $\mathbf{z}$ (yaw, direction)
2. Rotation about $\mathbf{y}$ (pitch, attitude)
3. Rotation about $\mathbf{x}$ (roll, bank)

$\mathbf{R}' = \mathbf{z} \mathbf{R} \mathbf{y} \mathbf{z} \mathbf{R} \mathbf{y} \mathbf{z}$

\[ \mathbf{Y}'(\Theta) = \begin{pmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{pmatrix} \]

\[ \mathbf{Z}'(\Psi) = \text{Same as on pg 5-10 (with } \Theta \text{ replaced by } \Psi \) 

In Quantum Mechanics and Particle Physics
Choose

1. Rotation about $\mathbf{z} = \mathbf{z} \phi$
2. Rotation about $\mathbf{y}' = \mathbf{y}' \theta$
3. Rotation about $\mathbf{z}'' = \mathbf{z}'' \psi$

Another set of parameters - Cayley-Klein Parameters

Skip
Euler's Theorem on the motion of a Rigid Body
with one point fixed:

"The general displacement of a rigid body with
one point fixed is a rotation about some fixed
axis."

Note: The axis must (of course) pass through the fixed pt.

As discussed previously, we can describe the motion
of a rigid body with one point fixed by an
orthogonal transformation matrix: \( \mathbf{A}(t) \)

\[
\mathbf{A}(0) = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Initial condition

Now suppose the axis of rotation is \( \mathbf{R} \).

Then \( \mathbf{R}' = \mathbf{A} \mathbf{R} \) but if Euler's theorem is
true then \( \mathbf{R}' = \mathbf{R} \). Thus, Euler's theorem may be
restated:

For any real orthogonal matrix specifying physical
motion of a rigid body with one fixed point, there
exists a vector \( \mathbf{R} \) such that

\[
\mathbf{A} \mathbf{R} = \mathbf{R}
\]

This is a special case of the general eigenvalue problem.
General eigenvalue problem:
\[ \overline{A} \mathbf{R} = \lambda \mathbf{R} \]  (1)

where \( \lambda \) is a constant, which may be complex.
The \( \lambda \)'s are called **eigenvalues** and the corresponding vectors \( \mathbf{R} \) are called **eigen vectors**.

Finally, we can restate Euler’s theorem:

**Non-trivial**

Any real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has one and only one eigenvalue \( +1 \).

---

**Note:** **Non-trivial means we are not interested in** \( \overline{A} = \mathbf{I} \). (Since in that case no rotation is necessary and any vector \( \mathbf{R} \) will work!)

---

Rewrite eqn (1) above

\[ (\overline{A} - \overline{\lambda} \mathbf{I}) \mathbf{R} = 0 \]

\[ \begin{align*}
(a_{11} - \lambda) x + a_{12} Y + a_{13} Z &= 0 \\
&
(a_{21} - \lambda) x + (a_{22} - \lambda) Y + a_{23} Z &= 0 \\
(a_{31} - \lambda) x + a_{32} Y + (a_{33} - \lambda) Z &= 0
\end{align*} \]

---

Three linear equations has non-trivial solution if

\[ \det (\overline{A} - \overline{\lambda} \mathbf{I}) = 0. \]

Or using our notation \[ |\overline{A} - \overline{\lambda} \mathbf{I}| = 0. \] → called the characteristic eqn. or secular equation.
The characteristic equation:

\[
\begin{vmatrix}
\tilde{A} - \lambda \tilde{I} \\
\end{vmatrix} =
\begin{vmatrix}
\tilde{a}_{11} - \lambda & \tilde{a}_{12} & \tilde{a}_{13} \\
\tilde{a}_{21} & \tilde{a}_{22} - \lambda & \tilde{a}_{23} \\
\tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} - \lambda \\
\end{vmatrix} = 0 = -\lambda^3 + B \lambda^2 + C \lambda + D
\]

The cubic characteristic equation has three solutions; or three eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and three corresponding eigenvectors \( x_1, x_2, x_3 \).

Can recast the whole eigenvalue-eigenvector problem in matrix form.

Define \( \tilde{X} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \) as \( X \).

Define \( \tilde{\lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \) as \( \lambda \).

Then, all three eigenvalue-eigenvector problems may be written in matrix form:

\[ \tilde{\lambda} \tilde{X} = \tilde{A} \tilde{X} \]

or \[ [A_{ij}X_{ij}] = [X_{ij} \lambda_{ij}] \]

For example:

\[ a_{11}x_i + a_{12}y_j + a_{13}z_j = \lambda_1 x_i \quad ; \quad i=1, j=1 \]
\[ a_{21}x_i + a_{22}y_j + a_{23}z_j = \lambda_2 y_i \quad ; \quad i=2, j=1 \]
\[ a_{31}x_i + a_{32}y_j + a_{33}z_j = \lambda_3 z_i \quad ; \quad i=3, j=1 \]
Thus, can multiply equation 1 by $\bar{X}^{-1}$ on the right by $\bar{X}^{-1}$:

$$\bar{X}^{-1} \bar{A} \bar{X} = \bar{\lambda} = \text{diagonal matrix}.$$ 

Earlier we defined a similarity transformation: $\bar{A}' = \bar{B} \bar{A} \bar{B}^{-1}$. Hence the similarity transformation $\bar{X}^{-1}$ will diagonalize the matrix $\bar{A}$.

Now back to Euler's theorem and its proof.

We want to show that (except for the trivial case of $\bar{A} = \bar{I}$) $\bar{A}$ has one and only one eigenvalue $\lambda = +1$.

(a) First will show that at least one $\lambda_i = +1$.

Consider

$$(\bar{A} - \bar{I}) \bar{X}^t = \bar{A} \bar{X}^t - \bar{X}^t = \bar{X}^t - \bar{X}^t = (\bar{X}^t - \bar{X}^t)^t$$

(remember $\bar{X}^t$ = transpose of $\bar{X}$)

and for orthogonal matrix

$$\bar{X}^t = \bar{X}^{-1}.$$ 

Take the determinant of both sides:

$$|\bar{A} - \bar{I}| |\bar{X}^t| = |(\bar{X}^t - \bar{X}^t)|$$

since if $\bar{A}$ represents a transformation of a rigid body $|\bar{A}| = +1$.

Thus

$$|\bar{A} - \bar{I}| = |\bar{X}^t - \bar{X}^t|$$

But if $\bar{B}$ is any $n \times n$ matrix then $|\bar{B}| = (-1)^n |\bar{B}|$.

In our case $n = 3$ and we have

$$|\bar{A} - \bar{I}| = (-1)^3 |\bar{A} - \bar{I}| = |\bar{A} - \bar{I}| = 0$$

$$\Rightarrow \lambda = +1.$$ 

Compare this last result with $|\bar{A} - \bar{\lambda} \bar{I}| = 0$. 

Thus at least one of the eigenvalues \( \lambda_i = +1 \).

(b) Now we show that given (a) then all eigenvalues must have unit magnitude:

From \( \mathbf{\tilde{X}}^{-1} \mathbf{\tilde{A}} \mathbf{\tilde{X}} = \mathbf{\tilde{X}} \)

we have

\[
|\mathbf{\tilde{X}}^{-1}| |\mathbf{\tilde{A}}| |\mathbf{\tilde{X}}| = |\mathbf{\tilde{X}}| = \lambda_1 \lambda_2 \lambda_3
\]

Or

\[
|\mathbf{\tilde{A}}| = \lambda_1 \lambda_2 \lambda_3 = +1 \quad (\text{since } |\mathbf{\tilde{A}}| = +1)
\]

But from (a) \( \lambda_3 \) (say) = +1

\[
\therefore \lambda_1 \lambda_2 = +1
\]

There are three possibilities:

1. \( \lambda_1 = \lambda_2 = +1 \)

2. \( \lambda_1 = \lambda_2 = -1 \)

3. Both \( \lambda_1 \) and \( \lambda_2 \) are complex nos.

   In this case \( \lambda_1 = \lambda_2^* \)

   and \( \lambda_1 \lambda_2 = +1 \Rightarrow |\lambda_1|^2 = +1 \)

   or \( \lambda_1 = e^{i\phi} \) and \( \lambda_2 = e^{-i\phi} \)

Thus, we have found the eigenvalues for any orthogonal matrix \( \mathbf{A} \) with \( \det \mathbf{A} = +1 \) must be

either 1. \( \lambda_1 = \lambda_2 = \lambda_3 = +1 \) \( \Rightarrow \) trivial case \( \mathbf{A} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

2. \( \lambda_3 = +1 \), \( \lambda_1 = -1 = \lambda_2 \)

   or

3. \( \lambda_3 = +1 \), \( \lambda_1 = e^{i\phi} \) \( \Rightarrow \lambda_2 = e^{-i\phi} \)

Hence, we have shown that, except for the trivial case of \( \mathbf{A} = \mathbf{I} \), there is one and only one eigenvalue of \( \mathbf{A} \) which is equal to +1. Hence, Euler's Theorem is proved.
How do we find the orientation of the axis of rotation?

Set \( \lambda = 1 \) and solve for \( x, y, \) and \( z \).

\[
\begin{align*}
(a_{11} - 1) x + a_{12} y + a_{13} z &= 0 \\
(a_{21} - 1)y + a_{22} x + a_{23} z &= 0 \\
(a_{31} - 1)z + a_{32} y + (a_{33} - 1) x &= 0
\end{align*}
\]

Homogeneous eqns.

Note the the length of vector \( \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) is arbitrary because of homogeneous eqns. (i.e., if \( \mathbf{x} \) is a solution then so is \( a \mathbf{x} \) for any \( a \).

Can remove this arbitary factor by requiring \( \mathbf{x} \) to be a unit vector:

\[
\mathbf{x} \cdot \mathbf{x} = 1 \quad \Rightarrow \quad x^2 + y^2 + z^2 = 1
\]

Then this "normalized" eigenvector is a unit vector which is directed along the axis of rotation.

How do we find the angle of rotation?

It is always possible to use a similarity transformation \( \mathbf{B} \) to transform to a coordinate system such that the \( z \)-axis in the new system is directed along the axis of rotation. Then the original transformation matrix \( \mathbf{A} \) may be expressed in the new coord. system:

\[
\mathbf{A}' = \mathbf{B} \mathbf{A} \mathbf{B}^{-1}
\]

But by Euler's theorem we know that \( \mathbf{A}' = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

Now we use the fact that the trace of a matrix is left unchanged by a similarity transformation.
Thus

\[ \text{Tr } \tilde{A}' = a'_{22} = 2 \cos \Theta + 1 \]

\[ \text{Tr } \tilde{A} \text{ since trace is invariant under similarity transformation.} \]

\[ (\text{We prove this below}) \]

\[ \therefore \cos \Theta = \frac{1}{2} \left[ \text{Tr } \tilde{A} - 1 \right]. \]

Note: If have all eigenvalues then \( \text{Tr } \tilde{A} = \text{Tr } \tilde{X} = \lambda_1 + \lambda_2 + \lambda_3 \)

\[ \lambda_1 + \lambda_2 + \lambda_3 = 1 + e^{i \phi} + e^{-i \phi} = 1 + 2 \cos \phi \]

Hence, \( \phi = 0 \).

---

Proof that trace of a matrix is invariant under a similarity transformation:

\[ \tilde{A}' = B \tilde{A} B^{-1} \]

\[ \{a'_{i,j}\} = b_{i,z} a_{z,m} b^{-1}_{m,j} \quad \text{(sum over } z = 1, 2, 3 \text{ and } m = 1, 2, 3) \]

Now \( \text{Tr } \tilde{A}' = b_{\mu,2} a_{2,m} b^{-1}_{m,1} \quad \text{sum over } z = 1, 2, 3; \mu = 1, 2, 3; \text{ and } \pi = 3 \).

Do the sum over \( \tau \) first:

\[ b^{-1}_{\mu,1} b_{\mu,2} = \tilde{B}^{-1} \tilde{B} = \tilde{I} = [\delta_{\mu,2}] \]

Thus the sum over \( \mu \) collapses

\[ \text{Tr } \tilde{A}' = a_{2,2} \delta_{\mu,2} = a_{2,2} = \text{Tr } \tilde{A}' \]

Q.E.D.
Example and Demonstration:

Find the equivalent rotation (axis and angle) corresponding to a rotation of $+90^\circ$ about the $\bar{z}$ axis and then $+90^\circ$ about the (new) $\bar{x}'$ axis (i.e. $\phi = \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ and $\psi = 0$)

\[
\begin{align*}
\bar{R}' &= \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{R} \\
\bar{R}'' &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}
\end{align*}
\]

\[
\bar{A} = \bar{A}_2 \bar{A}_1 = \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ +1 & 0 & 0 \end{pmatrix}
\]

\[\begin{array}{r}
\Rightarrow \\
\text{Tr} \bar{A} = 0 \Rightarrow \cos \theta = \frac{1}{2} [\text{Tr} \bar{A} - 1] = -\frac{1}{2}
\end{array}\]

or \[\theta = \pm 120^\circ\].

To find direction of axis:

\[
(\bar{A} - I) \bar{x} = 0
\]

\[
\begin{pmatrix} -1 & +1 & 0 \\ 0 & -1 & +1 \\ +1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\Rightarrow \begin{cases} 
-x + y = 0 \\
-y + z = 0 \\
-x - z = 0
\end{cases} \Rightarrow \begin{cases} 
x = y \\
y = z \\
x = z
\end{cases}
\]

\[x \cdot x = 1 = 3x^2 \Rightarrow x = \frac{1}{\sqrt{3}} \quad \bar{n} = \frac{1}{\sqrt{3}} (x + y + z)
\]

Demonstration using Rubik's cube.
Corollary to Euler's Theorem:

Chasle's Theorem: the most general displacement of a rigid body is a translation plus a simple rotation.

Infiniteesimal rotations behave (very nearly) like a vector directed along the rotation axis. (actually a pseudovector)

Finite rotations cannot be described by vectors since for example
\[ \vec{A} + \vec{B} = \vec{B} + \vec{A} \]
but we have already shown that \[ \vec{A} \cdot \vec{B} \neq \vec{B} \cdot \vec{A} \]
where \( \vec{A} \) and \( \vec{B} \) denote finite rotations.

However, the same objection does not apply if we limit ourselves to infinitesimal rotations.

If we rotate the axes by an infinitesimal amount about some axis then an arbitrary vector \( \mathbf{R} \) will be described in the rotated coord, system by
\[ \mathbf{R'} = \mathbf{R} + \hat{\mathbf{e}} \mathbf{R}, \]
where \( \hat{\mathbf{e}} = [\hat{\mathbf{e}}_{ij}] \) and \( \hat{\mathbf{e}}_{ij} \) are infinitesimals—meaning that a limit will be taken such that \( \hat{\mathbf{e}}_{ij} \to 0 \) and hence in successive operations only first order terms need to be retained.
Thus, for an infinitesimal transformation with one point fixed:

\[ R' = (\vec{I} + \lambda \vec{E}) R = \vec{A}_{de} R \]

1. **Two successive infinitesimal rotations commute.**

   Proof:
   \[ (\vec{I} + \lambda \vec{E}_1) (\vec{I} + \lambda \vec{E}_2) = (\vec{I} + \lambda \vec{E}_1 + \lambda \vec{E}_2 + \lambda \vec{E}_1 \vec{E}_2) \]
   \[ = (\vec{I} + \lambda \vec{E}_1, \lambda \vec{E}_2) = (\vec{I} + \lambda \vec{E}_1)(\vec{I} + \lambda \vec{E}_2) \text{ neglecting } O(\lambda^3) \]
   Done.

2. If \( \vec{A}_{de} = (\vec{I} + \lambda \vec{E}) \) then \( \vec{A}_{de}^{-1} = (\vec{I} - \lambda \vec{E}) \).

   Proof:
   \[ (\vec{I} + \lambda \vec{E})(\vec{I} - \lambda \vec{E}) = \vec{I} + \lambda \vec{E} - \lambda \vec{E} = \vec{I} \text{ (neglecting } O(\lambda^3) \text{)} \]
   Done.

3. \( \vec{E} \) is antisymmetric (or skew) matrix, i.e. \( E_{ij} = -E_{ji}; \) or \( \vec{E} = -\vec{E}^t \).

   Proof: \( \vec{A}_{de} \) must be an orthogonal matrix.

   Thus \( \vec{A}_{de}^t = \vec{A}_{de}^{-1} \)

   \[ (\vec{I} + \lambda \vec{E})^t = (\vec{I} + \lambda \vec{E}^t) = \vec{A}_{de}^{-1} = (\vec{I} - \lambda \vec{E}) \]

   \[ \therefore \vec{E}^t = -\vec{E} \text{ or } \vec{E} = -\vec{E}^t \text{. Done.} \]

Thus \( \vec{E} \) must be of the form

\[ \vec{E} = \begin{pmatrix}
0 & \lambda E_3 & \lambda E_2 \\
\lambda E_3 & 0 & \lambda E_1 \\
\lambda E_2 & -\lambda E_1 & 0
\end{pmatrix} \]

- diagonal elements must = 0
- only three other independent elements, \( E_1, E_2 \) and \( E_3 \).
Now consider an arbitrary infinitesimal rotation as consisting of three successive rotations about one of the body axes:

\[
\text{d}R_1 \text{ about } x_1 \text{ axis: } \quad \tilde{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(dR_1) & -\sin(dR_1) \\ 0 & \sin(dR_1) & \cos(dR_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -dR_1 & 1 \end{pmatrix}
\]

or \( \tilde{A}_1 = \mathbb{I} + \mathbf{L}_1 \text{d}R_1 \) where \( \mathbf{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \)

\[
\text{d}R_2 \text{ about } x_2 \text{ axis: } \quad \tilde{A}_2 = \begin{pmatrix} \cos(dR_2) & 0 & \sin(dR_2) \\ 0 & 1 & 0 \\ \sin(dR_2) & 0 & \cos(dR_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\text{d}R_2 \\ 0 & 1 & 0 \\ \text{d}R_2 & 0 & 1 \end{pmatrix}
\]

or \( \tilde{A}_2 = \mathbb{I} + \mathbf{L}_2 \text{d}R_2 \) where \( \mathbf{L}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \)

\[
\text{d}R_3 \text{ about } x_3 : \quad \tilde{A}_3 = \begin{pmatrix} \cos(dR_3) & \sin(dR_3) & 0 \\ -\sin(dR_3) & \cos(dR_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \text{d}R_3 & 0 \\ -\text{d}R_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

or \( \tilde{A}_3 = \mathbb{I} + \mathbf{L}_3 \text{d}R_3 \) where \( \mathbf{L}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \)

And

\[
\tilde{A}_e = \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 = \mathbb{I} + \mathbf{L}_1 \text{d}R_1 + \mathbf{L}_2 \text{d}R_2 + \mathbf{L}_3 \text{d}R_3 = \mathbb{I} + \tilde{\mathbf{d}E}
\]

Thus

\[
\tilde{\mathbf{d}E} = \begin{pmatrix} 0 & \text{d}R_3 & -\text{d}R_2 \\ -\text{d}R_3 & 0 & \text{d}R_1 \\ \text{d}R_2 & -\text{d}R_1 & 0 \end{pmatrix}
\]
Instantaneous angular velocity vector is

Now any vector \( \mathbf{R} \) viewed from the infinitesimally rotated coordinate system becomes

\[
\mathbf{R}' = \mathbf{R} + d\mathbf{R}
\]

where

\[
d\mathbf{R} = \mathbf{E} \cdot d\mathbf{R} = \begin{pmatrix} 0 & d\tau_3 & -d\tau_2 \\ -d\tau_3 & 0 & d\tau_1 \\ d\tau_2 & -d\tau_1 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}
\]

Expanding we have

\[
d\mathbf{R} = \begin{pmatrix} dR_1 \\ dR_2 \\ dR_3 \end{pmatrix} = \begin{pmatrix} R_2 d\tau_3 - R_3 d\tau_2 \\ R_3 d\tau_1 - R_1 d\tau_3 \\ R_1 d\tau_2 - R_2 d\tau_1 \end{pmatrix} = \mathbf{R} \times d\mathbf{S}_1
\]

where \( d\mathbf{S}_1 \) has vector components \( d\tau_1, d\tau_2, \) and \( d\tau_3 \).

\[
d\mathbf{R} = \mathbf{R} \times d\mathbf{S}_1
\]

\[
|d\mathbf{R}| = R d\mathbf{S}_1 \sin \theta
\]

and direction as shown.

Note: Positive \( d\mathbf{S}_1 \) rotates coordinate system according to r.h. rule — (or counter-clockwise view from arrow head)

"passive" viewpoint.

Thus if view from "active" viewpoint then the vector \( \mathbf{R} \) is rotated clockwise by \( |d\mathbf{S}_1| \) as shown above.

If the infinitesimal rotation \( d\mathbf{S}_1 \) occurs in time \( dt \), then define instantaneous angular velocity vector, \( \mathbf{\omega} \):

\[
\mathbf{\omega} = \frac{d\mathbf{S}_1}{dt} \quad \text{and} \quad \frac{d\mathbf{R}}{dt} = \frac{\mathbf{E}}{dt} \mathbf{R} = \mathbf{R} \times \mathbf{\omega}
\]
Expressing the angular velocity vector $\omega$ in terms of changes in Euler angles:

Consider a rigid body with instantaneous angular velocity vector $\omega$.

We can express $\omega$ in terms of the body-fixed axis and in terms of the rates of change in the Euler angles $\phi$, $\theta$, and $\psi$.

$$d\mathbf{L} = d\mathbf{L}_x + d\mathbf{L}_y + d\mathbf{L}_z$$

$$\omega = \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}_x}{dt} + \frac{d\mathbf{L}_y}{dt} + \frac{d\mathbf{L}_z}{dt}$$

We want to express $\frac{d\mathbf{L}}{dt} = \omega$ in the final body coordinate system $x''', y''', z'''$.

$$\frac{d\mathbf{L}_z}{dt} = \dot{\psi} \mathbf{e}_z'''''$$

already in $'''''$ system.

$$\frac{d\mathbf{L}_y}{dt} = \dot{\phi} \mathbf{e}_z'''$$

and

$$\frac{d\mathbf{L}_x}{dt} = \dot{\theta} \mathbf{e}_z''''$$

Need to express these vectors in $'''''$ system.

To get from $x', y', z'$ system to $'''''$ system need to first rotate about $\mathbf{e}_x'$ axis by $\theta$ and then about the new $\mathbf{e}_z''$ axis by $\psi$:

$$\mathbf{X'}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

$$\mathbf{X''}(\psi) = \begin{pmatrix} \cos \psi & \sin \psi \cos \theta & \sin \psi \sin \theta \\ -\sin \psi & \cos \psi \cos \theta & \cos \psi \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

$$(\mathbf{X''}(\psi))^{\mathbf{X'}(\theta)} = \begin{pmatrix} \cos \psi & \sin \psi \cos \theta & \sin \psi \sin \theta \\ -\sin \psi & \cos \psi \cos \theta & \cos \psi \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & \Theta \cos \phi & \Theta \sin \phi \sin \psi \\ 0 & -\Theta \sin \phi & \Theta \cos \phi \cos \psi \end{pmatrix}$$
And therefore adding $\frac{d\theta}{dt} = \psi \frac{\dot{z}}{\dot{z}}$ we have

$\omega$ expressed in the body-axis:

\[ \omega_x = \dot{\psi} \cos \phi + \dot{\phi} \sin \Theta \sin \psi \]
\[ \omega_y = -\dot{\psi} \sin \psi + \dot{\phi} \sin \Theta \cos \psi \]
\[ \omega_z = \dot{\phi} \cos \Theta + \dot{\psi} \]

Can also express $\omega$ in the original $x, y, z$ space axis:

\[ \omega_x = \frac{d\theta}{dt} + \frac{d}{dt} \left( \psi \frac{\dot{z}}{\dot{z}} \right) + \frac{\dot{z}}{\dot{z}} \frac{d}{dt} \left( \dot{z} \right) \]
\[ \omega_y = \frac{d\phi}{dt} \cos \Theta - \frac{d}{dt} \left( \psi \frac{\dot{z}}{\dot{z}} \right) \cos \psi \]
\[ \omega_z = \frac{d\Theta}{dt} + \dot{\psi} \cos \Theta \]

Resulting in:

\[ \omega_x = \dot{\phi} \cos \phi + \dot{\psi} \sin \Theta \sin \phi \]
\[ \omega_y = \dot{\phi} \sin \phi - \dot{\psi} \sin \Theta \cos \phi \]
\[ \omega_z = \dot{\phi} + \dot{\psi} \cos \Theta \]
Note: Transformation properties of \( dI \). (and hence of \( \omega = \frac{d\mathbf{L}}{dt} \))

**Defn:** A polar vector \( \mathbf{C} \) (ordinary vector) transforms under any orthogonal transformation \( \mathbf{B} \) as

\[
\mathbf{C}' = \mathbf{B} \mathbf{C} \quad \text{i.e.} \quad C'_i = B_{ij} C_j
\]

**Examples:**
- \( \mathbf{r} \) (position vector), \( \mathbf{F} = m \mathbf{r} \) (or any other time deriv. of \( \mathbf{r} \) )

- An axial vector: \( \mathbf{L} \) (pseudo-vector) transforms under any orthogonal transformation \( \mathbf{B} \) as

\[
\mathbf{L}' = (\det \mathbf{B}) \mathbf{B} \mathbf{L} \quad \text{i.e.} \quad L'_i = (\det \mathbf{B}) B_{ij} L_j
\]

**Note:**
- Proper rotations \( \Rightarrow \det \mathbf{B} = +1 \)
- Improper rotations (inversions) \( \Rightarrow \det \mathbf{B} = -1 \)

For a simple inversion, \( B_{ij} = -S_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \)

- Polar vectors: \( \mathbf{C} \to \mathbf{C}' = -\mathbf{C} \) reverse in sign.
- Pseudo vectors: \( \mathbf{L} \to \mathbf{L}' = \mathbf{L} \) do not reverse remain unchanged.

Examples of pseudo-vectors:
- i) cross product of two polar vectors: \( \mathbf{A} = \mathbf{C} \times \mathbf{D} \)
- ii) angular momentum: \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \)
- iii) magnetic field vector: \( \mathbf{B} \)
- iv) instantaneous angular velocity vector: \( \omega = \frac{d\mathbf{L}}{dt} \)

For inversion:
\[
\frac{d\mathbf{E}}{dt} = \mathbf{E}^* = \begin{bmatrix} 0 & -\partial_2 \omega \partial_3 \partial_1 \\ -\partial_3 \omega \partial_1 \partial_2 & 0 \\ -\partial_1 \omega \partial_3 \partial_2 & 0 \end{bmatrix}
\]

\[
\mathbf{E}' = \mathbf{E}^* \mathbf{E} = \begin{bmatrix} 0 & -\partial_2 \omega \partial_3 \partial_1 \\ -\partial_3 \omega \partial_1 \partial_2 & 0 \\ -\partial_1 \omega \partial_3 \partial_2 & 0 \end{bmatrix}
\]

**Note:** In inverted co-ord system cross product defined as \( \mathbf{a} \times \mathbf{b} \) is still an axial vector, obeys left hand rule.
Proof that $dS$ is a pseudo-vector: (See Goldstein Appendix C)

Want to find out how $dS$ transforms under orthogonal trans., $\tilde{B}$.

From definition of $dS$:

$$d\tau = \begin{bmatrix} 0 & d\tau_y & -d\tau_z \\ -d\tau_z & 0 & d\tau_x \\ d\tau_x & -d\tau_y & 0 \end{bmatrix} \Rightarrow d\tau_{mn} = e_{mni} d\tau_i$$

Let $\tilde{B}$ be any orthogonal trans. matrix.

Then

$$d\tau' = \tilde{B} \tau \tilde{B}^{-1}$$

or

$$d\tau'_{kl} = b_{kl} d\tau_{mn} b^{-1}_{nl} = b_{kl} e_{mni} b^{-1}_{ni} d\tau_i$$

since $\tilde{B}^{-1} = \tilde{B}^t$.

Now we use the following expression for the determinant of a $3 \times 3$ matrix:

$$\det \tilde{A} = \varepsilon_{ijk} a_{pi} a_{qj} a_{r}$$

(alt. $\det \tilde{A} = \varepsilon_{ijk} a_{pi} a_{qj} a_{r}$)

for any choice of $m, n, j$ which is a cyclic permutation of $1, 2, 3$.

and

$$\det \tilde{A} = -\varepsilon_{ijk} a_{pi} a_{qj} a_{r}$$

and $\varepsilon_{ijk} a_{pi} a_{qj} a_{r} = 0$ if any two of $m, n, j$ are equal.

So we can write this as

$$e_{mni} \det \tilde{A} = \varepsilon_{ijk} a_{pi} a_{qj} a_{r}$$

And if $\tilde{A}$ is an orthog. matrix $\tilde{B}$, then $\det \tilde{B} = \pm 1 \Rightarrow (\det \tilde{B})^2 = +1$

and

$$e_{mni} = \varepsilon_{ijk} b_{pi} b_{qj} b_{r} (\det \tilde{B})$$

[alt.: $e_{mni} = \varepsilon_{ijk} b_{pi} b_{qj} b_{r} (\det \tilde{B})$]
Now use this back in the expression for \( \mathbf{e}'_{kj} \):

\[
d\mathbf{e}'_{kl} = b_{km} b_{ln} d\mathbf{e}_{mn} = b_{km} b_{ln} \varepsilon_{mnp} d\Sigma_j .
\]

\[
d\mathbf{e}'_{kl} = \varepsilon_{kl} i d\Sigma_i = b_{km} b_{ln} \varepsilon_{pji} b_{pm} b_{nj} b_{ij} (\det \mathbf{B}) d\Sigma_j .
\]

But, because \( \mathbf{B} \) is orthog., \( b_{km} b_{pm} = \delta_{kp} \) and \( b_{ln} b_{nj} = \delta_{lj} \).

\[
\therefore \quad d\mathbf{e}'_{kl} = \varepsilon_{kl} i d\Sigma_i = \varepsilon_{kl} i b_{ij} d\Sigma_j (\det \mathbf{B})
\]

\[
\therefore \quad \text{can identify } d\Sigma_i = (\det \mathbf{B}) b_{ij} d\Sigma_j
\]

Q.E.D.

\[
\begin{bmatrix}
  d\Sigma_1 \\
  d\Sigma_2 \\
\end{bmatrix}
\]

transforms like a vector under proper rotations \( \mathbf{B} \) but not under improper rotations \( \mathbf{B}^{-1} \).

Note: ordinary cross product of two polar vectors is a pseudo-vec.

\[
\begin{aligned}
\mathbf{B} &\quad \mathbf{C}' = \mathbf{B} \mathbf{C} \quad \mathbf{D}' = \mathbf{B} \mathbf{D} \\
\alpha'_{m} &\quad \varepsilon_{mnj} \alpha'_{n} \alpha'_{j} = \varepsilon_{mnj} b_{mj} c_{nj} b_{ij} d_{kl} \\
\alpha'_{m} &\quad \varepsilon_{pqk} b_{mp} b_{q} b_{k} (\det \mathbf{B}) b_{nj} b_{mk} c_{ij} d_{kl} \\
\alpha'_{m} &\quad = b_{mp} \varepsilon_{pqk} c_{kl} d_{kl} (\det \mathbf{B}) = (\det \mathbf{B}) b_{mp} \alpha_p \quad \text{Q.E.D.}
\end{aligned}
\]
Rate of change of a vector:

The change (in time $dt$) of any arbitrary vector, $\mathbf{G}$, as viewed from the body axes will differ from the corresponding change as viewed from the space axes because of the effect of the rotation of the body axes.

$$d\mathbf{G}_{\text{body}} = d\mathbf{G}_{\text{space}} + d\mathbf{G}_{\text{Rot}}$$

But we have just shown that (pg 5-23)

$$d\mathbf{G}_{\text{Rot}} = \mathbf{G} \times d\mathbf{\Omega}.$$

Thus

$$d\mathbf{G}_{\text{space}} = d\mathbf{G}_{\text{body}} - d\mathbf{G}_{\text{Rot}} = d\mathbf{G}_{\text{body}} + d\mathbf{\Omega} \times \mathbf{G}.$$  

Or dividing through by the infinitesimal time element:

$$\frac{d\mathbf{G}}{dt}_{\text{space}} = \frac{d\mathbf{G}}{dt}_{\text{body}} + \mathbf{\omega} \times \mathbf{G},$$

where $\mathbf{\omega} = \frac{d\mathbf{\Omega}}{dt} = \text{instantaneous angular rate of rotation of the rigid body}.$

This applies to any vector:

$$\frac{d}{dt} \mathbf{G}_{\text{space}} = \frac{d}{dt} \mathbf{G}_{\text{body}} + \mathbf{\omega} \times \mathbf{G}.$$
Motion in a rotating reference frame:

Often the motion of a particle is viewed from a coordinate system which is actually rotating. For example, motion of particle on the earth's surface.

→ Assume the origin of "space" system and "body" system are the same point.

Let it describe the position of a particle with respect to an origin at center of the rotating system (i.e. the earth).

For example:

"space" system is ref. frame (a inertial) with origin at center of earth and direction space - fixed.

"body" system is ref. frame fixed in the earth and rotating with it with avg. vel. \( \omega = \text{const.} \) (assumption).

\[
\frac{\text{d}r}{\text{d}t} \bigg|_{\text{space}} = \frac{\text{d}r}{\text{d}t} \bigg|_{\text{body}} + \omega \times r
\]

or

\[
\mathbf{v}_s = \mathbf{v}_b + \omega \times r,
\]

Now operating on \( \mathbf{v}_s \) we find

\[
\frac{\text{d} \mathbf{v}_s}{\text{d}t} \bigg|_{\text{space}} = \frac{\text{d} \mathbf{v}_s}{\text{d}t} \bigg|_{\text{body}} + \omega \times \mathbf{v}_s = \frac{d}{dt} \left( \mathbf{v}_b + \omega \times r \right) + \omega \times \left( \mathbf{v}_b + \omega \times r \right)
\]

\[
\frac{d \mathbf{v}_s}{dt} \bigg|_{\text{space}} = \frac{d \mathbf{v}_b}{dt} \bigg|_{\text{body}} + \frac{d \mathbf{w}}{dt} \bigg|_{\text{body}} \times r + \omega \times \frac{d \mathbf{v}_b}{dt} \bigg|_{\text{body}} + \omega \times \mathbf{v}_b + \omega \times (\omega \times r)
\]

If we assume \( \frac{d \mathbf{w}}{dt} \bigg|_{\text{body}} = 0 \) then we have

(Note: \( \frac{d \mathbf{v}_s}{dt} \bigg|_{\text{space}} = \frac{d \mathbf{v}_s}{dt} \bigg|_{\text{body}} \))

\[
\mathbf{a}_s = \mathbf{a}_b + \omega \times \mathbf{v}_b + \omega \times (\omega \times r)
\]
Thus, there are two "extra" terms in the acceleration:

Assuming the "space" frame to be an inertial frame:

\[ m\alpha_s = F_{\text{tot}} \]

Viewed from the body frame:

\[ \alpha_b = \alpha_s - 2 \mathbf{\omega} \times \mathbf{V}_b - \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

or

\[ m\alpha_b = m\alpha_s - 2m (\mathbf{\omega} \times \mathbf{V}_b) - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

Finally:

(For the Earth:

\[ \Theta \equiv \text{latitude}; \quad \Theta_c \equiv \text{co-latitude} \]

\[ \phi \equiv \text{longitude}; \quad \omega_{\text{Earth}} \approx 7.3 \times 10^{-5} \text{ rad/sec} \]

First consider the centrifugal "force"

\[ \mathbf{\omega} \times \mathbf{r} = \omega r \sin \Theta \text{ directed } \perp \text{ to } \mathbf{\omega} \]

and \( \perp \) to \( \mathbf{r} \)

\[ |\mathbf{\omega} \times \mathbf{r}| = \omega r \sin \Theta = v \]

velocity tangential to the circular path which a point fixed in the rigid body traverses due to the rotation \( \mathbf{\omega} \).

\[ \mathbf{F}_{\text{centrifugal}} = m \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) = m\omega^2 \mathbf{r} \perp \text{ directed outward from the circular path,} \]

\( \equiv \) opposite the centripetal force in space system.
On the Physical origin of the Coriolis "force".

The Coriolis "force":  

\[ \mathbf{F}_{\text{Coriolis}} = -2m \mathbf{\omega} \times \mathbf{v}_b \]

In considering the physical origin of the Coriolis force, it is useful to consider the components of \( \mathbf{v}_b \) along the three orthogonal directions shown:

\[ \hat{\mathbf{e}}_t = \text{unit vector directed tangent to circle at point } r. \text{ (directed along } \mathbf{\omega} \times \hat{\mathbf{e}} \text{)} \]

\[ \hat{\mathbf{e}}_l = \text{unit vector outward from cir.} \text{ (directed along } -\mathbf{\omega} \times (\mathbf{\omega} \times \hat{\mathbf{e}}) \text{)} \]

and

\[ \hat{\mathbf{e}}_w = \text{unit vector directed } \perp \text{ to } \mathbf{\omega}. \]

\[ \therefore \mathbf{v}_b = v_{\parallel b} \hat{\mathbf{e}}_t + v_{\perp b} \hat{\mathbf{e}}_l + v_{\omega b} \hat{\mathbf{e}}_w \text{ and } \mathbf{\omega} = \omega \hat{\mathbf{e}}_w. \]

\[ \mathbf{F}_{\text{Coriolis}} = -2m\omega v_{\parallel b} (\hat{\mathbf{e}}_w \times \hat{\mathbf{e}}_l) - 2m\omega v_{\parallel b} (\hat{\mathbf{e}}_w \times \hat{\mathbf{e}}_t) - 2m\omega v_{\perp b} \omega (\hat{\mathbf{e}}_w \hat{\mathbf{e}}_w) \]

or

\[ \mathbf{F}_{\text{Coriolis}} = -2m\omega v_{\parallel b} \hat{\mathbf{e}}_t + 2m\omega v_{\perp b} \hat{\mathbf{e}}_l \]

This (perhaps more interesting) part of the Coriolis force is directed \( \perp \) to the Centrifugal force. This part of the Coriolis force is simply a "correction" to the Centrifugal force.
Physical origin of Coriolis force continued:

\[ \mathbf{F}_{\text{Coriolis}} = -2m \mathbf{\omega} \times \mathbf{v}_b = -2m \omega \mathbf{v}_b \mathbf{e}_\theta + 2m \omega \mathbf{v}_b \mathbf{e}_1 \]

The \( 2m \omega \mathbf{v}_b \mathbf{e}_1 \) term is directed along the direction of the Centrifugal force (i.e., along \( \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \)) and is a correction to the Centrifugal force due to the fact that if \( \mathbf{v}_b \neq 0 \) then the tangential speed of the point is not simply \( |\mathbf{\omega} \times \mathbf{r}| \) but \( |\mathbf{\omega} \times \mathbf{r} + \mathbf{v}_b \mathbf{e}_1| \) and the rotation of the coord. system (see diagram) directly leads to

\[ m \frac{\Delta \mathbf{v}}{\Delta t} \mathbf{e}_1 = m \frac{\mathbf{v}_b \omega \Delta t}{\Delta t} \mathbf{e}_1 \]

Add 1 and 2 to get \( 2m \omega \mathbf{v}_b \mathbf{e}_1 \)

The \(-2m \omega \mathbf{v}_b \mathbf{e}_1 \) term is directed \( \perp \) to Centrifugal force.

It has two contributions also:

1. Part due directly to the rotation of the body coord. while \( \mathbf{v}_b \) remains stationary: contributes

\[ m \frac{\Delta \mathbf{v}}{\Delta t} (\mathbf{e}_1) = -m \mathbf{v}_b \omega \mathbf{e}_1 \]

2. Part due to change in radius and point on the rigid body axis at larger radius have larger tangential vel. Hence, the vel. will tend to lag behind if it moves to larger radius. Contributes

\[ -m \frac{\Delta \mathbf{v}_b}{\Delta t} \mathbf{e}_1 = -m \omega \frac{\Delta \mathbf{r}_b}{\Delta t} \mathbf{e}_1 = -m \omega \mathbf{v}_b \mathbf{e}_1 \]

Add 1 and 2 to get \(-2m \omega \mathbf{v}_b \mathbf{e}_1 \).
What if the origin of the body frame is moving?

We have assumed (pg. 530) the origins of "space" and "body" systems coincide. Now we remove this assumption.

\[ O' \text{ is origin of "body" system $\{x', y', z'\}$ coods.} \]

\[ O' = \text{position of pcl. relative to origin of "body" system.} \]

\[ \Gamma = \Gamma R + \Gamma' \]

\[ V_s = \frac{dR}{dt}_s + \frac{dR'}{dt}_s = V'_0 + V'_s \]

\[ a_s = \frac{dV_s}{dt}_s = A'_0 + \frac{dV'_s}{dt}_s = A'_0 + a'_s \]

As before, we can express $a'$ in terms of the body (primed) coord. system:

\[ a'_s = a'_0 + 2\omega \times V'_0 + \omega \times (\omega \times \Gamma') \]

And, since the space system is an inertial frame:

\[ m\ddot{a}_s = F \]

Combine 1, 2 and 3 and solve for $m\ddot{a}_0'$:

\[ m\ddot{a}_0' = F - m\dot{A}_0' - 2m(\omega \times V_0') - m\omega \times (\omega \times \Gamma') \]

Extra "force" term because $O'$ is accel.
Tidal effect due to the sun.

Note: For the case of the earth moving about the sun while rotating on its axis it is very nearly correct to neglect both $A_0$ and $F_{\text{sun}}$ since they very nearly cancel.

i.e. if point is at center of earth
then $F = mA_0$ and they cancel exactly.

\[
\begin{align*}
M_0 A_0 &= -\frac{GM_0 M_1 \hat{R}}{R_{\text{ES}}} \Rightarrow A_0 = -\frac{GM_1}{R_{\text{ES}}} \hat{R} \\
F_{\text{sun}} &= -\frac{GM_1 M_2 \hat{R}}{R_{\text{ES}}} = mA_0,
\end{align*}
\]

Thus, the only term which survives $F_{\text{sun}} - mA_0$ is the difference between the gravitational attraction of the sun at the center of earth and at the point $R'$. These are the so called solar tidal effects and depend on the gradient in the gravitational attraction of the sun at the position of the earth.

There are similar tidal effect due to the moon and the motion of the earth-moon system.
Motion in coord. system embedded in the earth:

Assume space coord. system with origin at center of earth
(Neglect motion of earth around sun -- only tidal effects)

(Note: \( \omega_e \) due earth spinning on axis \( \approx \frac{2\pi}{24 \text{ hr}} \times \frac{1 \text{ hr}}{3600 \text{ s}} \approx 7.3 \times 10^{-5} \text{ rad/sec.} \)

and

\( \Omega_e \) due earth orbit about sun \( \sim \frac{\omega_e}{365} \approx 2 \times 10^{-7} \text{ rad/sec.} \)

\[
\mathbf{m} \mathbf{a}_{\text{space}} = \mathbf{F}_g + m \mathbf{g}(r) = m \mathbf{a}_b + 2m \omega_e \times \mathbf{V}_b + m \omega_e \times (\omega_e \times \mathbf{r})
\]

or

\[
m \mathbf{a}_b = \mathbf{F}_g + m \mathbf{g}(r) - m \omega_e \times (\omega_e \times \mathbf{r}) - 2m (\omega_e \times \mathbf{V}_b)
\]

where \( \mathbf{r} \) is meas. from the center of the earth.

Thus

\[
m \mathbf{a}_b = \mathbf{F}_g + m \mathbf{g}_e - 2m (\omega_e \times \mathbf{V}_b)
\]

where

\[
\mathbf{g}_e = \mathbf{g}_e - \omega_e \times (\omega_e \times \mathbf{r})
\]

A plumb bob held at colatitude \( \theta_c \) will hang along \( \mathbf{g}_e \) (since \( \mathbf{V}_b = 0 \)).

Hence, in general a Plumb bob does not point to the mass center of the earth.

But actually this effect is quite small since it is of order \( \omega_e^2 \).

(More correctly should compare \( \omega_e^2 \) with \( g \))

\[
\omega_e^2 R_e \approx 10^{-2} \text{ m/s}^2 \ll 10 \text{ m/s}^2
\]

\( R_e \approx 6.4 \times 10^6 \text{ m} \)

\( \omega_e R_e \approx 7.8 \times 10^4 \approx 87 \text{ m/s} \)
The Foucault Pendulum:

First consider a pendulum set up in a laboratory at the north pole. Then it is clear when viewed from the "space" system that the pendulum, if it is started swinging in a plane, that it will simply continue to swing in the same plane. Now, viewed from the laboratory frame which rotates with the earth, we will see the plane of the pendulum precess with angular velocity \( \Omega = -\omega_e \), i.e. the plane of the pendulum will precess backwards (as compared to the earth's rotation) 2\( \pi \) radians per day. This precession of the plane of a pendulum is easy to understand when we are at the pole. Now, consider the same pendulum set up in a laboratory at colatitude \( \theta_c \).

Consider the coord. system fixed in the earth with \( z' \) axis along
\[
\mathbf{e}_c = \mathbf{e} - \omega_e \times (\mathbf{e} \times \mathbf{e}_e).
\]

Thus, when hanging undisturbed the pendulum hangs directly along the \( z' \) axis.

*Note: The main point here is that the \( \omega_e \times (\mathbf{e} \times \mathbf{e}_e) \) term is a constant correction to \( \mathbf{e} \).
Now if we give the bob a small velocity, then a Coriolis force will act which causes the plane of the pendulum to precess. Suppose we assume the pendulum precesses at a constant angular velocity $\Omega(\theta)$. We want to find $\Omega(\theta)$ and test the assumption at the same time.

If the assumption is true, then if we get into a new coord. system rotating about $z'$ with angular velocity $\Omega(\theta)$, we will simply see the pendulum swinging in a constant plane. Hence, in this new coord. system there must be no Coriolis force in the plane of motion of the pendulum. Thus, if we start pendulum moving along $x''$-axis, then there will be no Coriolis force along the $y''$-axis.

$$m \mathbf{\alpha}_{\text{coriolis}} = -2m (\mathbf{\Omega}'' \times \mathbf{v}_b)$$

$$\mathbf{\Omega}'' = \mathbf{\Omega}_0 + \Omega(\theta) \mathbf{z}' = \omega_{x''} \mathbf{x}'' + \omega_{y''} \mathbf{y}'' + \omega_{z''} \mathbf{z}''$$

$$\mathbf{v}_b = v_{x''} \mathbf{x}''$$

$$\therefore m \mathbf{\alpha}_{\text{coriolis}} = -2m \omega_{x''} v_{x''} \mathbf{z}'' + 2m \omega_{y''} v_{x''} \mathbf{z}''$$

Thus we want $\omega_{x''} v_{x''} = 0$ for any $v_{x''}$.

$$\omega_{x''} = \omega_v \cos \Theta_v + \Omega(\theta)$$

$$\therefore \Omega(\theta) = -\omega_v \cos \Theta_v$$

Plane precesses of osc. clockwise in Northern Hemisphere and counter clockwise in Southern Hemisphere.