Added note on the change of co-ords, from \( \Gamma_1, \Gamma_2 \) to \( \Gamma \) and \( \Gamma' \).

\[
\Gamma_1 = \Gamma_2 + \Gamma' \quad \text{definition of } \Gamma
\]

\[
M \cdot \Gamma = m_1 \Gamma'_1 + m_2 \Gamma'_2 \quad \text{definition of center of mass }
\]

\[
\Gamma_1 = \Gamma + \Gamma' \quad \text{giving the position of m}_1 \text{ w.r.t. the c.m.}
\]

\[
\Gamma_2 = \Gamma + \Gamma' \quad \text{giving the position of m}_2 \text{ w.r.t. the c.m.}
\]

Notes:

\[
m_1 \Gamma_1' + m_2 \Gamma_2' = 0
\]

Thus, we can define \( \mu \) such that \[
\begin{align*}
m_1 \Gamma_1' &= \mu \Gamma' \\
\mu &= \frac{m_2}{m_1 + m_2}
\end{align*}
\]

Then \[
m_2 \Gamma_2' = -\mu \Gamma
\]

\( \mu \) is called "the reduced mass" or "the effective mass."

And since \[
\Gamma = \Gamma_1' - \Gamma_2'
\]

we have \[
\Gamma = \frac{1}{\mu} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \Gamma \\
\frac{1}{\mu} = \left( \frac{1}{m_1} + \frac{1}{m_2} \right)
\]

Important note on the physical interpretation of results:

We show that the general two-body problem reduces to

the motion of the center of mass \( \Gamma(t) \) — easy to solve

Plus the motion of the effective mass \( \mu \) in a

at position \( \Gamma \) moving in central potential \( V(r) \).

This motion reduces to motion in a plane.

So the solution is for \( r(t), \theta(t) \).

\[
V(r) \quad \text{Physically we want } \Gamma_1' \text{ & } \Gamma_2'
\]

the motion of \( m_1 \) & \( m_2 \) relative to the C.M.

But since \[
\begin{align*}
\Gamma_1' &= \mu \Gamma' \\
\Gamma_2' &= -\mu \Gamma
\end{align*}
\]

the motion of \( m_1 \) r.t. c.m is just given by

\[
\frac{r'}{m_1} = \frac{\mu}{m_2} \theta = \theta_c \quad , \text{i.e. only a scale factor } \frac{\mu}{m_2}.
Summary of properties for orbits in central potential \( V(r) = -\frac{k}{r} \):

- \( k > 0 \) attractive
- \( k < 0 \) repulsive

Orbit eqn. gives general orbit:

\[
\begin{align*}
  r(\theta) &= \frac{C}{1 + \varepsilon \cos \theta} \quad C = \frac{l^2}{\mu} \quad \varepsilon^2 = 1 - 2\frac{k^2 \mu^2}{l^2} \\
  E &= \text{energy of orbit} \quad E = \frac{\mu r^2}{2l^2} (E^2 - 1) \\
  \varepsilon &= \text{eccentricity} \\
  r_1 &= \text{closest approach} = \frac{C}{1 + \varepsilon} \\
  \mu &= \frac{m_1 m_2}{m_1 + m_2} > 0
\end{align*}
\]

For \( k > 0 \) (attractive force) \( \Rightarrow C > 0 \):

- \( E \) always \( \geq -\frac{\mu R^2}{2l^2} \)
- \( \Rightarrow \) bound orbits

\[ \begin{align*}
  R > 0 & \quad -\frac{\mu R^2}{2l^2} \leq E < 0 \quad \Rightarrow 0 \leq \varepsilon < 1 \quad \Rightarrow \text{bound orbits} \\
  f = \frac{1}{e} & \quad r_1 = r_p = \text{periapsis} \\
  a = c(1 - e^2) & \quad b = c(1 - e^2)^{1/2} \\
  \text{semi-major axis} & \quad \text{semi-minor axis}
\end{align*} \]

Bound orbits are ellipses:

- orbit is a parabola

\[ E = 0 \quad \Leftrightarrow \quad E = 1 \quad (E = 0 \Rightarrow \dot{r}_p = \dot{r}_0 = 0) \]

- orbit is a hyperbola (attractive force, hyperbola)

For \( k < 0 \) (repulsive force) \( C < 0 \) \( r = r_1 \) at \( \theta = 0 \) \( \Rightarrow \varepsilon > 1 \):

\[ \begin{align*}
  E > 0 \quad |E| > 1 & \quad \text{together with } r = r_1 \text{ at } \theta = 0 \quad \Rightarrow \varepsilon > 1 \\
  \cos \theta & \Rightarrow -\frac{1}{\varepsilon} < \theta < 0 \\
  \quad \cos \theta_c = -\frac{1}{\varepsilon} \\
  r \rightarrow \infty & \text{ at } \pm \theta_c, \quad |\theta_c| > \frac{\pi}{2}
\end{align*} \]

orbit is a hyperbola (repulsive force, hyperbola)

\[ \begin{align*}
  E > 0 & \quad \text{always} \\
  1 - |E| \cos \theta & \leq 0 \quad \Rightarrow -\theta_c < \theta < \theta_c \\
  |E| \cos \phi & \geq \frac{1}{|E|} \quad |\phi_c| < \frac{\pi}{2} \\
  \text{orbit is a hyperbola (repulsive force, hyperbola)}
\end{align*} \]
\[ V(r) = -\frac{k}{r} \]

For a given \( k \neq 0 \) and \( E > 0 \) we will get a hyperbolic orbit. Either the attractive \((k > 0)\) or the repulsive \((k < 0)\) hyperbola.

\[ \cos \theta = \frac{1}{e} \quad e > 1 \]

\[ \cos \phi = \frac{1}{|\epsilon|} \quad |\epsilon| > 1 \quad \epsilon < -1 \]

**AttrACTIVE:**
\[ \theta \] goes from \(-\theta_c \) to \(+\theta_c \) giving \(+l > 0\)

**Repulsive:**
\[ \theta \] goes from \(-\phi_c \) to \(+\phi_c \) giving \(+l > 0\)

Impact parameter: \( s \Rightarrow l = \mu \frac{v_{\infty}}{c} \quad s = \mu \frac{v_{\infty}}{c} \quad s = \frac{1}{l} \]

\[ E = \frac{1}{2} \mu \frac{v_{\infty}^2}{c} = \frac{1}{2} \mu \frac{v^2}{c} \quad \text{because} \quad V = 0 \]

\[ \text{or} \quad v_{\infty} = \sqrt{\frac{2E}{\mu}} \quad \text{when} \quad r = \infty \]

\[ s = \sqrt{2EM} = \frac{1}{2E} \frac{\sqrt{2}}{cm} \frac{\sqrt{E}}{m} \]

Since \( E = \frac{\mu k^2}{2\epsilon^2} (\epsilon^2 - 1) \)

\[ s = \frac{|k|}{2E} \sqrt{\epsilon^2 - 1} \quad \text{or} \quad s = \frac{\epsilon^2 - 1}{E} \]

Suppose \( l, E \) \( |k| \) same in both cases. \( |\epsilon| \) and \( 1/|\epsilon| \) are the same.

But the distance of closest approach will not be the same in the two cases.

\[ \Gamma_{k>0} = \frac{|\epsilon|}{1+|\epsilon|} < \Gamma_{k<0} = \frac{|\epsilon|}{|\epsilon|-1} \quad |\epsilon| > 1 \]